

# Spectral instability for the complex Airy operator and even non-selfadjoint anharmonic oscillators

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## Abstract

We study the instability of the spectrum for a class of non-selfadjoint anharmonic oscillators, estimating the behavior of the instability indices (*i. e.* the norm of spectral projections) associated with the large eigenvalues of these oscillators. More precisely, we consider the operators

$$\mathcal{A}(m, \theta) = -\frac{d^2}{dx^2} + e^{i\theta}|x|^m$$

defined on  $L^2(\mathbb{R})$ , with

$$|\theta| < \min \left\{ \frac{(m+2)\pi}{4}, \frac{(m+2)\pi}{2m} \right\},$$

in the case where  $m$  is either equal to 1 or an even positive integer. We get asymptotic expansions for the instability indices, extending the results of [6] and [8].

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**Key words :** non-selfadjoint, complex WKB method, eigenvalues and eigenfunctions expansions, completeness of eigenfunctions.

## 1 Introduction

It has been known for several years that the spectrum of a non-selfadjoint operator  $\mathcal{A}$ , acting on an Hilbert space  $\mathcal{H}$ , can be very unstable under small perturbations of  $\mathcal{A}$ . In other words, unlike in the selfadjoint case, the norm of the resolvent of  $\mathcal{A}$  near the spectrum can blow up much faster than the inverse distance to the spectrum. Equivalently, the spectrum of its perturbations  $\mathcal{A} + \varepsilon\mathcal{B}$ , with  $\varepsilon > 0$  and any  $\mathcal{B} \in \mathcal{L}(\mathcal{H})$ ,  $\|\mathcal{B}\| \leq 1$ , is not necessarily included in the set  $\{z \in \mathbb{C} : d(z, \sigma(\mathcal{A})) \leq \varepsilon\}$ . Let  $\lambda \in \sigma(\mathcal{A})$  be an isolated eigenvalue of  $\mathcal{A}$ , and let  $\Pi_\lambda$  denote the spectral

projection associated with  $\lambda$ . In order to understand the instability of  $\lambda$  (in the above sense), we define the *instability index* of  $\lambda$  as the number

$$\kappa(\lambda) = \|\Pi_\lambda\|,$$

see [6].

Of course  $\kappa(\lambda) \geq 1$  in any case, and  $\kappa(\lambda) = 1$  when  $\mathcal{A}$  is selfadjoint.

If  $\Pi_\lambda$  has rank 1, that is, if  $\lambda$  is simple in the sense of the algebraic multiplicity, we have a convenient expression for  $\kappa(\lambda)$ , which we shall use extensively in the following: if  $u$  and  $u^*$  denote respectively eigenvectors of  $\mathcal{A}$  and  $\mathcal{A}^*$  associated with  $\lambda$  and  $\bar{\lambda}$ , one can easily check [4] that

$$\kappa(\lambda) = \frac{\|u\|\|u^*\|}{|\langle u, u^* \rangle|}. \quad (1.1)$$

To understand the relation between spectral instability and instability indices, we denote by  $\sigma_\varepsilon(\mathcal{A})$  the  $\varepsilon$ -pseudospectra of  $\mathcal{A}$ , that is the family of sets, indexed by  $\varepsilon$ ,

$$\sigma_\varepsilon(\mathcal{A}) = \left\{ z \in \rho(\mathcal{A}) : \|(\mathcal{A} - z)^{-1}\| > \frac{1}{\varepsilon} \right\} \cup \sigma(\mathcal{A}).$$

From the perturbative point of view,  $\sigma_\varepsilon(\mathcal{A})$  can be seen as the union of the perturbed spectra, in the following sense:

$$\sigma_\varepsilon(\mathcal{A}) = \bigcup_{\substack{\mathcal{B} \in \mathcal{L}(L^2), \\ \|\mathcal{B}\| \leq 1}} \sigma(\mathcal{A} + \varepsilon\mathcal{B}).$$

This equivalent formulation follows from a weak version of a theorem due to Roch and Silbermann [24].

Instability indices are closely related to the size of  $\varepsilon$ -pseudospectra around  $\lambda$ . Indeed, let  $\sigma_\varepsilon^\lambda$  denote the connected component of  $\sigma_\varepsilon(\mathcal{A})$  containing  $\lambda$ , and assume for simplicity that  $\sigma_\varepsilon^\lambda \cap \sigma(\mathcal{A}) = \{\lambda\}$  and  $\sigma_\varepsilon^\lambda$  is bounded. Let  $\partial\sigma_\varepsilon^\lambda$  and  $|\partial\sigma_\varepsilon^\lambda|$  denote respectively the boundary and the perimeter of  $\sigma_\varepsilon^\lambda$ . Then, taking into account the expression of the spectral projection,

$$\Pi_\lambda = \frac{1}{2i\pi} \int_{\partial\sigma_\varepsilon^\lambda} (z - \mathcal{A})^{-1} dz,$$

we get [4]

$$\kappa(\lambda) \leq \frac{1}{2\pi} \int_{\partial\sigma_\varepsilon^\lambda} \|(\mathcal{A} - \xi)^{-1}\| d\xi \leq \frac{1}{2\pi\varepsilon} |\partial\sigma_\varepsilon^\lambda|.$$

Hence,

$$|\partial\sigma_\varepsilon^\lambda| \geq 2\pi\varepsilon\kappa(\lambda). \quad (1.2)$$

In the finite dimensional setting at least, instability indices give a better description of pseudospectra. If  $\mathcal{A} \in \mathcal{M}_n(\mathbb{C})$  is a diagonalizable matrix with distinct eigenvalues  $\lambda_1, \dots, \lambda_n$ , Embree and Trefethen show [28] that the  $\varepsilon$ -pseudospectra

are rather well approximated by disks of radius  $\varepsilon\kappa(\lambda_k)$  around the eigenvalues. More precisely, there exists  $\varepsilon_0 > 0$  such that, for all  $\varepsilon \in ]0, \varepsilon_0[$ ,

$$\bigcup_{\lambda_k \in \sigma(\mathcal{A})} D(\lambda_k, \varepsilon\kappa(\lambda_k) + \mathcal{O}(\varepsilon^2)) \subset \sigma_\varepsilon(\mathcal{A}) \subset \bigcup_{\lambda_k \in \sigma(\mathcal{A})} D(\lambda_k, \varepsilon\kappa(\lambda_k) + \mathcal{O}(\varepsilon^2)). \quad (1.3)$$

In the case of an infinite dimensional space, the validity of this statement should be investigated, as well as the dependance on  $\lambda_k$  of the  $\mathcal{O}(\varepsilon^2)$  terms.

In the following, we will consider some *anharmonic oscillators*

$$\mathcal{A}(m, \theta) = -\frac{d^2}{dx^2} + e^{i\theta}|x|^m, \quad (1.4)$$

where

$$|\theta| < \min \left\{ \frac{(m+2)\pi}{4}, \frac{(m+2)\pi}{2m} \right\}. \quad (1.5)$$

These operators are defined on  $L^2(\mathbb{R})$  by considering, first on  $\mathcal{C}_0^\infty(\mathbb{R})$ , the associated quadratic form, which is sectorial if  $\theta$  satisfies (1.5), see [6]. As stated in [6], its spectrum consists of a sequence of discrete simple eigenvalues, denoted in nondecreasing modulus order by  $\lambda_n = \lambda_n(m, \theta)$ ,  $|\lambda_n| \rightarrow +\infty$ , and the associated instability indices will be denoted by  $\kappa_n(m, \theta)$ .

All the spectral projections of  $\mathcal{A}(m, \theta)$  are of rank 1 (see Lemma 5 of [6]), and if  $u_n$  denotes an eigenfunction associated with  $\lambda_n(m, \theta)$ , then formula (1.1) yields

$$\kappa_n(m, \theta) = \frac{\int_{\mathbb{R}} |u_n(x)|^2 dx}{\left| \int_{\mathbb{R}} u_n^2(x) dx \right|}, \quad (1.6)$$

since in this case we have  $\mathcal{A}^*\Gamma = \Gamma\mathcal{A}$  where  $\Gamma$  denotes the complex conjugation, and thus  $u_n^* = \bar{u}_n$ .

The following pictures, which were performed by using a code inspired by [27], represent the boundaries of pseudospectra for some anharmonic oscillators (1.4), for several small values of  $\varepsilon$ . The inequality (1.2) and the inclusions (1.3) show that the values of  $\kappa_n(m, \theta)$  are related to the size, depending on  $\varepsilon$ , of these small disks appearing around the first eigenvalues on the simulations.

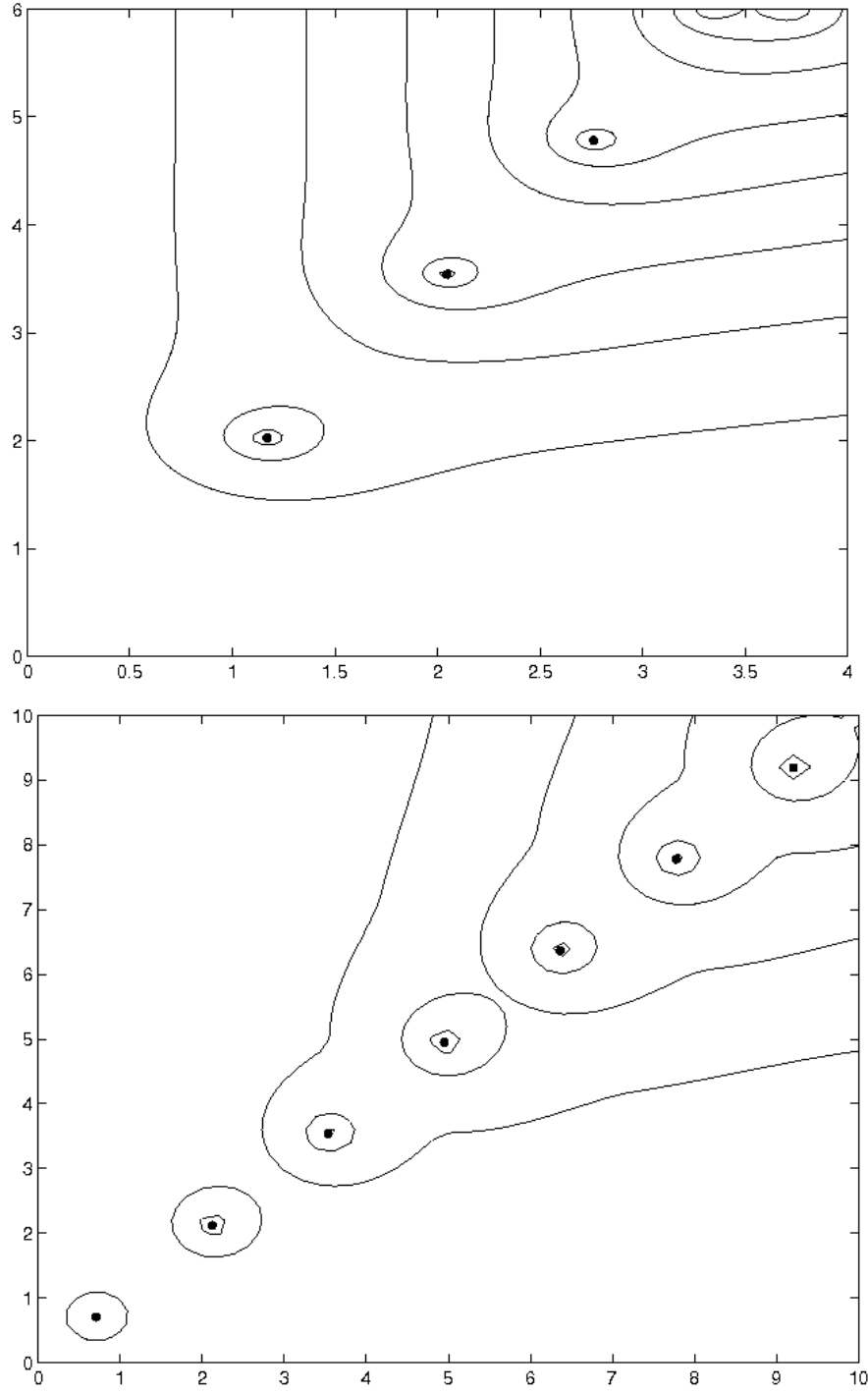


Figure 1:  $\varepsilon$ -Pseudospectra of complex Airy operator  $-\frac{d^2}{dx^2} + ix$  and harmonic oscillator  $-\frac{d^2}{dx^2} + ix^2$ .

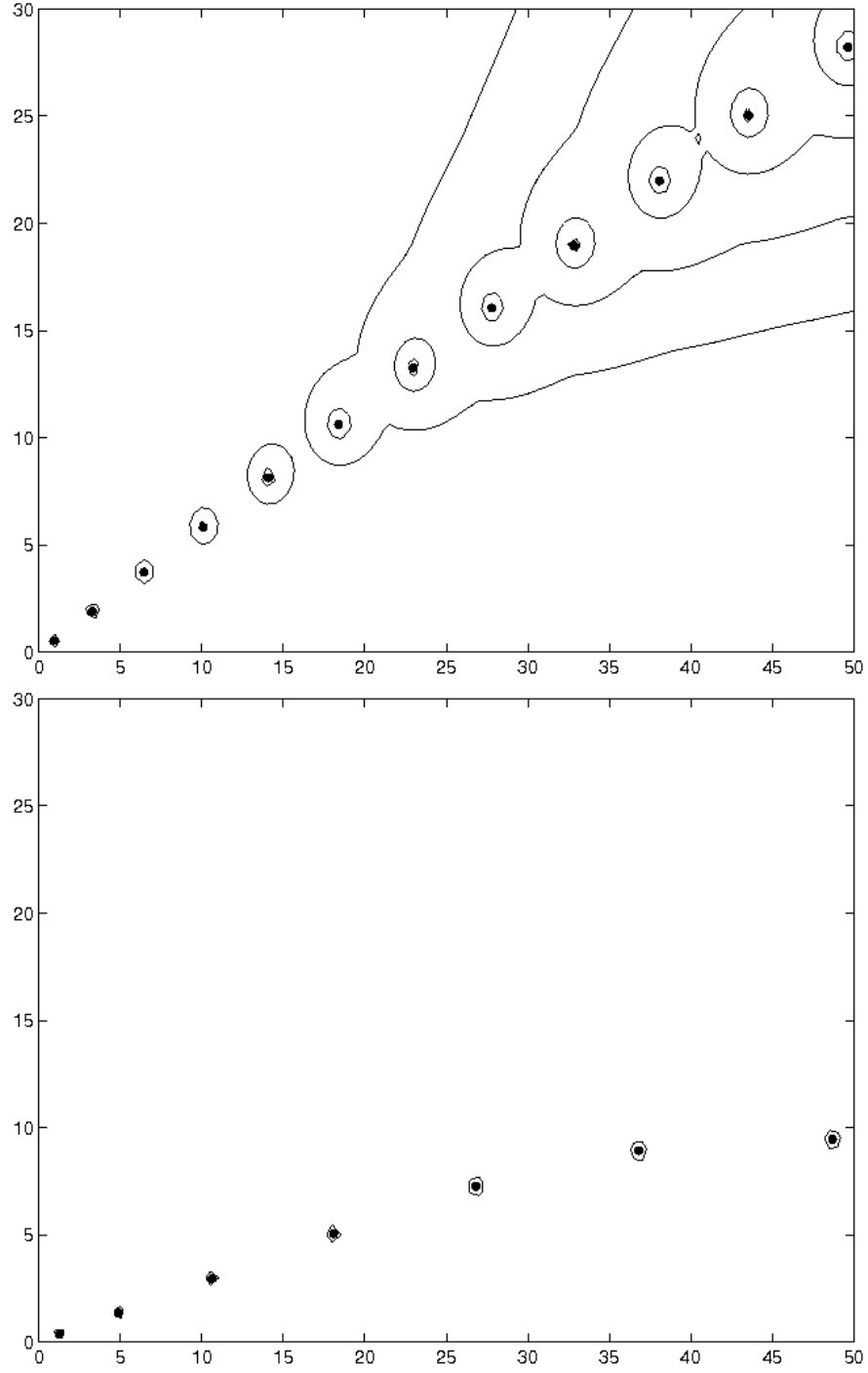


Figure 2: Pseudospectra of the complex quartic oscillator ( $k = 2$ ) and oscillator  $-\frac{d^2}{dx^2} + ix^{10}$ .

E.-B. Davies showed in [6] and [7] that  $\kappa_n(m, \theta)$  grows as  $n \rightarrow +\infty$  faster than any power of  $n$ , that is for all  $m \in ]0, +\infty[$  and  $\theta \neq 0$  satisfying (1.5), for all  $\alpha > 0$ , there exists  $N = N(\alpha, m, \theta) \in \mathbb{N}$  such that

$$\forall n \geq N, \quad \kappa_n(m, \theta) \geq n^\alpha.$$

This statement has been improved in the case  $m = 2$  of the harmonic oscillator (sometimes referred as the Davies operator), since E.-B. Davies and A. Kuijlaars showed [8] that  $\kappa_n(2, \theta)$  grows exponentially fast as  $n \rightarrow +\infty$ , with an explicit rate  $c(\theta)$  :

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \kappa_n(2, \theta) = c(\theta). \quad (1.7)$$

The purpose of our work is to prove that this last statement actually holds for the so-called *complex Airy operator*  $\mathcal{A}(1, \theta)$  and for the even anharmonic oscillators  $\mathcal{A}(2k, \theta)$ ,  $k \geq 1$ . More precisely, we will improve the estimate by getting asymptotic expansions in powers of  $n^{-1}$  as  $n \rightarrow +\infty$ .

Let us stress that such a growth of the instability indices implies that the family of eigenfunctions of  $\mathcal{A}(m, \theta)$  can not possess any of the “good” properties usually expected. It does not form a basis, neither in the Hilbert sense nor in the Riesz or Schauder sense (see [6], [20]). This excludes any hope of decomposing properly an  $L^2$  function along the eigenspaces of the operator.

We will prove in section 4, however, that the eigenfunctions form complete sets of the  $L^2$  space (Theorem 1.4).

Before stating the results of our work, let us specify some notation. Given two functions  $f, g$  and a real sequence  $(\alpha_j)_{j \geq 0}$ , we will write

$$f(\tau) \underset{\tau \rightarrow +\infty}{\sim} g(\tau) \sum_{j=0}^{+\infty} \alpha_j \tau^{-j}$$

to mean that, for all  $N \geq 1$ ,

$$f(\tau) = g(\tau) \left( \sum_{j=0}^N \alpha_j \tau^{-j} + \mathcal{O}(\tau^{-N-1}) \right)$$

as  $\tau \rightarrow +\infty$ .

We define likewise the symbol  $\underset{\tau \rightarrow 0}{\sim}$ .

The following results were announced in [18].

We focus on the cases  $m = 1$  and  $m = 2k$ ,  $k \in \mathbb{N}^*$  in (1.4), the first case corresponding to the complex Airy operator  $-\frac{d^2}{dx^2} + e^{i\theta}|x|$  studied in [3] and [14]. We will see in section 2 that this operator can be decomposed into the direct sum of its Dirichlet and Neumann realizations on  $\mathbb{R}^+$ , and we will prove the

**Theorem 1.1** Let  $(\kappa_n^D(1, \theta))_{n \geq 1}$  (resp.  $(\kappa_n^N(1, \theta))_{n \geq 1}$ ) denote the instability indices of the Dirichlet (resp. Neumann) realization of the operator

$$-\frac{d^2}{dx^2} + e^{i\theta}x \quad (1.8)$$

on  $\mathbb{R}^+$ , with  $0 < |\theta| < 3\pi/4$ .

There exist real sequences  $(\alpha_j(\theta))_{j \geq 1}$  and  $(\beta_j(\theta))_{j \geq 1}$  such that

$$\kappa_n^D(1, \theta) \underset{n \rightarrow +\infty}{\sim} \frac{K(\theta)}{\sqrt{n}} \exp(C(\theta)(n - 1/4)) \left(1 + \sum_{j=1}^{+\infty} \alpha_j(\theta) n^{-j}\right) \quad (1.9)$$

and

$$\kappa_n^N(1, \theta) \underset{n \rightarrow +\infty}{\sim} \frac{K(\theta)}{\sqrt{n}} \exp(C(\theta)(n - 3/4)) \left(1 + \sum_{j=1}^{+\infty} \beta_j(\theta) n^{-j}\right), \quad (1.10)$$

where

$$C(\theta) = 2\pi m_\theta^{3/2} |\sin \theta| \quad \text{and} \quad K(\theta) = \frac{m_\theta^{1/4}}{2\sqrt{3} |\sin \theta|},$$

with

$$m_\theta = \sqrt{1 + \frac{\sin^2(2\theta/3)}{\sin^2 \theta} - 2 \frac{\cos(\theta/3) \sin(2\theta/3)}{\sin \theta}} > 0.$$

Notice that, in the case of the complex Airy operator  $-\frac{d^2}{dx^2} + ix$  studied in [3] and [14], we get

$$\kappa_n^D(1, \pi/2) \underset{n \rightarrow +\infty}{\sim} \frac{1}{2^{3/4} \sqrt{3}} \left(1 + \sum_{j=1}^{+\infty} \alpha_j(\pi/2) n^{-j}\right) \exp\left(\frac{\pi}{\sqrt{2}}(n - 1/4)\right).$$

Regarding even anharmonic oscillators, we have a similar statement:

**Theorem 1.2** Let  $k \in \mathbb{N}^*$  and  $\theta$  be such that  $0 < |\theta| < \frac{(k+1)\pi}{2k}$ . If  $\kappa_n(2k, \theta)$  denotes the  $n$ -th instability index of  $\mathcal{A}(2k, \theta) = -\frac{d^2}{dx^2} + e^{i\theta}x^{2k}$ , then there exist  $K(2k, \theta) > 0$  and a real sequence  $(C^j(2k, \theta))_{j \geq 1}$  such that

$$\kappa_n(2k, \theta) \underset{n \rightarrow +\infty}{\sim} \frac{K(2k, \theta)}{\sqrt{n}} e^{c_k(\theta)n} \left(1 + \sum_{j=1}^{+\infty} C^j(2k, \theta) n^{-j}\right), \quad (1.11)$$

as  $n \rightarrow +\infty$ , with

$$c_k(\theta) = \frac{2(k+1)\sqrt{\pi}\Gamma\left(\frac{k+1}{2k}\right)\varphi_{\theta,k}(x_{\theta,k})}{\Gamma\left(\frac{1}{2k}\right)} > 0, \quad (1.12)$$

where

$$x_{\theta,k} = \left( \frac{\tan(|\theta|/(k+1))}{\sin(k|\theta|/(k+1)) + \cos(k|\theta|/(k+1)) \tan(|\theta|/(k+1))} \right)^{\frac{1}{2k}} \quad (1.13)$$

$$\varphi_{\theta,k}(x) = \operatorname{Im} \int_0^{xe^{i\frac{\theta}{2(k+1)}}} (1-t^{2k})^{1/2} dt. \quad (1.14)$$

**Remark 1.3** In the harmonic case  $k = 1$  (Davies operator), the first term in (1.11) yields the Davies-Kuijlaars theorem [8] :

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|\Pi_n\| = c_1(\theta) = 4\varphi_1 \left( \frac{1}{\sqrt{2 \cos(\theta/2)}} \right) = 2 \operatorname{Re} f \left( \frac{e^{i\theta/4}}{\sqrt{2 \cos(\theta/2)}} \right)$$

where  $f(z) = \log(z + \sqrt{z^2 - 1}) - z\sqrt{z^2 - 1}$ .

We are also interested in the properties of the eigenfunctions of the operator  $\mathcal{A}(m, \theta)$ . The following theorem has been proved in [3] in the case of Airy operator  $\mathcal{A}^D(1, \theta)$ , and in [6] in the harmonic oscillator case, as well as for  $\mathcal{A}(2k, \theta)$ ,  $k \geq 2$ ,  $|\theta| < \frac{\pi}{2}$ . We extend the result to any operator  $\mathcal{A}(2k, \theta)$  with  $|\theta| < \frac{(k+1)\pi}{2k}$  :

**Theorem 1.4** For all  $m = 1, 2k$ ,  $k \geq 1$ , and all  $\theta$  satisfying (1.5), the eigenfunctions of  $\mathcal{A}(m, \theta)$  form a complete set of the space  $L^2(\mathbb{R})$ .

Theorem 1.4 and the previous estimates enable us to study the convergence of the operator series defining the semigroup  $e^{-t\mathcal{A}(m, \theta)}$  associated with  $\mathcal{A}(m, \theta)$  when decomposed along the projections  $\Pi_n$ .

The following statement extends the result of [8] in the harmonic oscillator case.

**Corollary 1.5** Let  $|\theta| \leq \pi/2$  and  $e^{-t\mathcal{A}(m, \theta)}$  be the semigroup generated by  $\mathcal{A}(m, \theta)$ ,  $\lambda_n = \lambda_n(m, \theta)$  the eigenvalues of  $\mathcal{A}(m, \theta)$ , and  $\Pi_n = \Pi_n(m, \theta)$  the associated spectral projections.

Let  $T(\theta) = c_1(\theta)/\cos(\theta/2)$ , where  $c_1(\theta)$  is the constant in (1.12). The series

$$\Sigma_{m, \theta}(t) = \sum_{n=1}^{+\infty} e^{-t\lambda_n(m, \theta)} \Pi_n(m, \theta)$$

is not normally convergent in cases  $m = 1$  for any  $t > 0$ , and  $m = 2$  for  $t < T(\theta)$  ; in cases  $m = 2$  for  $t > T(\theta)$ , and  $m = 2k$  for any  $t > 0$ ,  $k \geq 2$ , the series converges normally towards  $e^{-t\mathcal{A}(m, \theta)}$  and, for  $N$  sufficiently large and for some constants  $C_1 = C_1(k, \theta)$  and  $C_2 = C_2(\theta)$ , the following estimate holds:

$$\|e^{-t\mathcal{A}(m, \theta)}(I - \Pi_{<N})\| \leq \begin{cases} \frac{C_1}{\sqrt{N}} e^{c_k(\theta)n} \exp(-t \operatorname{Re} \lambda_N), & k \geq 2 \\ \frac{C_2}{\sqrt{N}} \exp(-2 \cos(\theta/2)(t - T(\theta))N), & k = 1, t > T \end{cases} \quad (1.15)$$

where  $\Pi_{<N} = \Pi_1 + \dots + \Pi_{N-1}$  denotes the projection on the first  $N-1$  eigenspaces.

The paper is organized as follows. Section 2 is devoted to the analysis of the complex Airy operator and the proof of Theorem 1.1. We then handle the case of even anharmonic oscillators in section 3, while section 4 is independent and dedicated to the proofs of Theorem 1.4 and Corollary 1.5.

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## 2 The complex Airy operator

In this section we study the instability indices of the operator

$$\mathcal{A}(1, \theta) = -\frac{d^2}{dx^2} + e^{i\theta}|x|$$

acting on  $L^2(\mathbb{R})$ , with  $|\theta| < 3\pi/4$ . The potential  $e^{i\theta}|x|$  is singular at  $x = 0$ , which will lead us to decompose the operator along its action on the even and odd functions of  $L^2(\mathbb{R})$ .

### 2.1 Decomposition on the half-line

We use the decomposition

$$L^2(\mathbb{R}) = \mathfrak{P} \oplus \mathfrak{I}, \quad (2.1)$$

where  $\mathfrak{P}$  and  $\mathfrak{I}$  denote respectively the set of even and odd functions of  $L^2(\mathbb{R})$ . Thus we can identify a function  $u \in L^2(\mathbb{R})$  to a unique couple  $(u_P, u_I) \in \mathfrak{P} \times \mathfrak{I}$  such that  $u = u_P + u_I$ , where

$$u_P(x) = \frac{1}{2}(u(x) + u(-x))$$

is the even component of  $u$  and

$$u_I = \frac{1}{2}(u(x) - u(-x))$$

its odd component.

Let us now consider the restrictions of  $u_P$  and  $u_I$  to  $\mathbb{R}^+$ , still denoted  $u_P$  and  $u_I$ , and let  $\mathcal{A}^D(1, \theta)$  and  $\mathcal{A}^N(1, \theta)$  be respectively the Dirichlet and Neumann realizations on  $\mathbb{R}^+$  of the operator  $-\frac{d^2}{dx^2} + e^{i\theta}x$ . Then, a function  $u \in L^2(\mathbb{R})$  belongs to the domain of  $\mathcal{A}(1, \theta)$  if and only if  $u_P \in \mathcal{D}(\mathcal{A}^D(1, \theta))$  and  $u_I \in \mathcal{D}(\mathcal{A}^N(1, \theta))$ , even and odd functions satisfying respectively the Neumann and Dirichlet conditions at  $x = 0$ . Hence the decomposition (2.1) allows to identify  $\mathcal{A}(1, \theta)$  to the operator

$$\begin{pmatrix} \mathcal{A}^D(1, \theta) & 0 \\ 0 & \mathcal{A}^N(1, \theta) \end{pmatrix}$$

acting on  $\mathcal{D}(\mathcal{A}^D(1, \theta)) \times \mathcal{D}(\mathcal{A}^N(1, \theta))$ . In particular,

$$\sigma(\mathcal{A}(1, \theta)) = \sigma(\mathcal{A}^D(1, \theta)) \cup \sigma(\mathcal{A}^N(1, \theta)) \quad (2.2)$$

and the sequence of instability indices of  $\mathcal{A}(1, \theta)$  is the union of those of  $\mathcal{A}^D(1, \theta)$  and  $\mathcal{A}^N(1, \theta)$ , labelling the eigenvalues  $(\lambda_n)_{n \geq 1}$  of  $\mathcal{A}(1, \theta)$  such that  $|\lambda_j| \leq |\lambda_{j+1}|$ . The following statement (see [3]) describes the eigenfunctions and eigenvalues of  $\mathcal{A}^D(1, \theta)$  and  $\mathcal{A}^N(1, \theta)$ . The definition and main properties of the Airy function will be recalled in the next paragraph.

### Proposition 2.1

1. The spectrum of  $\mathcal{A}^D(1, \theta)$  is discrete and consists of the

$$\lambda_j^D = -e^{\frac{2i\theta}{3}} \mu_j, \quad j \in \mathbb{N},$$

where  $\mu_j$ ,  $j \geq 1$ , are the zeroes of the Airy function  $Ai$ ,  $\mu_1 < 0$  and  $\mu_{j+1} < \mu_j$ ,  $j \geq 1$ .

2. The spectral projection  $\Pi_n$  associated with  $\lambda_n^D$  has rank 1.
3. For all  $n \geq 1$ ,  $\int_{\mathbb{R}^+} u_n^2(x) dx \neq 0$  and

$$\kappa_n^D(1, \theta) = \frac{\int_{\mathbb{R}^+} |u_n(x)|^2 dx}{\left| \int_{\mathbb{R}^+} u_n^2(x) dx \right|}, \quad (2.3)$$

where

$$u_n(x) = Ai(\mu_n + x e^{i\theta/3}) \quad (2.4)$$

is an eigenfunction of  $\mathcal{A}^D(1, \theta)$  associated with the eigenvalue  $\lambda_n^D$ .

The analog result holds for  $\mathcal{A}^N(1, \theta)$ , replacing the zeroes  $\mu_n$  of the Airy function by its critical points  $\nu_n$ , which satisfy  $\nu_1 < 0$  and, for all  $n \geq 2$ ,  $\mu_n < \nu_n < \mu_{n-1}$ . According to (2.2), if  $(\lambda_n)_{n \geq 1}$  denote the eigenvalues of  $\mathcal{A}(1, \theta)$  (in increasing modulus order), then  $\lambda_{2k} = e^{2i\theta/3} |\mu_k|$  and  $\lambda_{2k+1} = e^{2i\theta/3} |\nu_{k+1}|$ . Furthermore,  $\kappa_{2k}(1, \theta) = \kappa_k^D(1, \theta)$  and  $\kappa_{2k+1}(1, \theta) = \kappa_{k+1}^N(1, \theta)$ .

Our work will be based, to a large extent, on the properties of the Airy function, which we recall in the next paragraph.

## 2.2 The Airy function

The Airy function  $Ai$  is the exponentially decaying solution as  $x \rightarrow +\infty$  of the (real) Airy equation

$$Ai''(x) = x Ai(x), \quad (2.5)$$

which is normalised by the condition

$$Ai(0) = \frac{1}{3^{2/3} \Gamma(\frac{2}{3})}.$$

Notice that  $Ai(x) > 0$  and  $Ai'(x) < 0$  for all  $x \geq 0$ . The Airy function can be extended in the whole complex plane as an holomorphic function, and its behavior at infinity is described by asymptotic expansions (see [1]), as well as the behavior of its derivative, zeroes and critical points.

In the following, we choose the principal value of the square root. The asymptotic properties of the Airy function are summarized in the following proposition, and will be used in the next paragraph to compute the instability indices  $\kappa_n^D(1, \theta)$ .

**Proposition 2.2** *There exist  $c_k, d_k, a_k, b_k, k \geq 1$ , such that*

1. *For any  $\alpha \in ]0, \pi]$ ,*

$$Ai(z) \underset{|z| \rightarrow +\infty}{\sim} \frac{1}{2\sqrt{\pi}z^{1/4}} e^{-\frac{2}{3}z^{3/2}} \left( 1 + \sum_{k=1}^{+\infty} (-1)^k c_k \left(\frac{2}{3}z^{3/2}\right)^{-k} \right) \quad (2.6)$$

*for  $|\arg z| \leq \pi - \alpha$  ;*

2.

$$\begin{aligned} Ai(-z) \underset{|z| \rightarrow +\infty}{\sim} & \frac{1}{\sqrt{\pi}z^{1/4}} \left( \sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) \left( 1 + \sum_{k=1}^{+\infty} (-1)^k c_{2k} \left(\frac{2}{3}z^{3/2}\right)^{-2k} \right) \right. \\ & \left. - \cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) \sum_{k=0}^{+\infty} (-1)^k c_{2k+1} \left(\frac{2}{3}z^{3/2}\right)^{-2k-1} \right) \end{aligned} \quad (2.7)$$

*for  $|\arg z| < \frac{2\pi}{3}$  ;*

3. *For any  $\alpha \in ]0, \pi]$ ,*

$$Ai'(z) \underset{|z| \rightarrow +\infty}{\sim} -\frac{z^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{3/2}} \left( 1 + \sum_{k=1}^{+\infty} (-1)^k d_k \left(\frac{2}{3}z^{3/2}\right)^{-k} \right) \quad (2.8)$$

*for  $|\arg z| \leq \pi - \alpha$  ;*

4.

$$\begin{aligned} Ai'(-z) \underset{|z| \rightarrow +\infty}{\sim} & -\frac{z^{1/4}}{\sqrt{\pi}} \left( \cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) \left( 1 + \sum_{k=1}^{+\infty} (-1)^k d_{2k} \left(\frac{2}{3}z^{3/2}\right)^{-2k} \right) \right. \\ & \left. + \sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) \sum_{k=0}^{+\infty} (-1)^k d_{2k+1} \left(\frac{2}{3}z^{3/2}\right)^{-2k-1} \right) \end{aligned} \quad (2.9)$$

*for  $|\arg z| < \frac{2\pi}{3}$  ;*

5.

$$|\mu_n| \underset{n \rightarrow +\infty}{\sim} \left( \frac{3\pi}{2}(n-1/4) \right)^{2/3} \left( 1 + \sum_{k=1}^{+\infty} (-1)^{k+1} a_k (n-1/4)^{-2k} \right) ; \quad (2.10)$$

6.

$$|\nu_n|_{n \rightarrow +\infty} \sim \left( \frac{3\pi}{2}(n - 3/4) \right)^{2/3} \left( 1 + \sum_{k=1}^{+\infty} (-1)^k b_k (n - 3/4)^{-2k} \right) ; \quad (2.11)$$

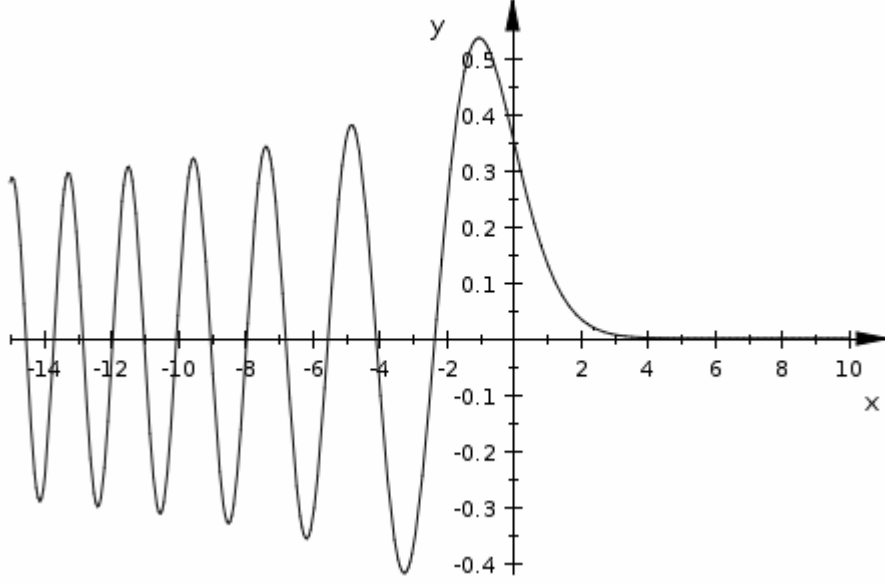


Figure 3: Graph of the Airy function  $Ai$  on  $\mathbb{R}$ .

We also recall the following useful lemma:

**Lemma 2.3** *The function*

$$F : x \mapsto xAi^2(x) - Ai'^2(x) \quad (2.12)$$

*is a primitive for  $Ai^2$ .*

**Proof:** Use equation (2.5):

$$\begin{aligned} F'(x) &= Ai^2(x) + 2xAi(x)Ai'(x) - 2Ai''(x)Ai'(x) \\ &= Ai^2(x) + 2Ai'(x)\underbrace{(xAi(x) - Ai''(x))}_{=0}. \end{aligned} \quad \square$$

We are now ready to compute the asymptotics of the instability indices for  $\mathcal{A}^D(1, \theta)$  and  $\mathcal{A}^N(1, \theta)$ .

### 2.3 Instability indices

We restrict our study to the Dirichlet realization  $\mathcal{A}^D(1, \theta)$  of the complex Airy operator, the case of the Neumann realization being similar, if we take care of replacing the zeroes  $\mu_n$  by the critical points  $\nu_n$ , and the expansion (2.10) by (2.11). We also assume  $\theta > 0$  without loss of generality (in the case  $\theta < 0$ , all the computations and statements below hold with  $|\theta|$  instead of  $\theta$ ).

Let  $u_n(x) = Ai(\mu_n + e^{i\theta/3}x)$  be the  $n$ -th eigenfunction of  $\mathcal{A}^D(1, \theta)$ , and let

$$I_n = \int_{\mathbb{R}^+} |u_n(t)|^2 dt, \quad J_n = \left| \int_{\mathbb{R}^+} u_n(t)^2 dt \right|$$

be respectively the numerator and denominator of expression (2.3). We will determine asymptotic expansions as  $n \rightarrow +\infty$  for both integrals.

#### Step 1: the $I_n$ term

Let  $\varepsilon > 0$  be fixed and small enough (see further specifications on the size of  $\varepsilon$ ). Then the decomposition

$$I_n = I_{n,\varepsilon} + R_{n,\varepsilon}, \quad (2.13)$$

where

$$I_{n,\varepsilon} := \int_{\varepsilon|\mu_n|}^{+\infty} |u_n(t)|^2 dt \quad \text{and} \quad R_{n,\varepsilon} := \int_0^{\varepsilon|\mu_n|} |u_n(t)|^2 dt,$$

will allow us to apply (2.6) and (2.7) uniformly in each integral  $R_{n,\varepsilon}$  and  $I_{n,\varepsilon}$  respectively.

**Step 1.a.** We first handle the  $I_{n,\varepsilon}$  term. According to (2.6), there exists a sequence  $(c'_k)_{k \geq 1}$  such that, for all  $N \geq 1$ ,

$$\begin{aligned} I_{n,\varepsilon} &= \int_{\varepsilon|\mu_n|}^{+\infty} \frac{1}{4\pi|\mu_n + te^{i\theta/3}|^{1/2}} \exp\left(-\frac{4}{3}\operatorname{Re}\left[(\mu_n + te^{i\theta/3})^{3/2}\right]\right) \\ &\quad \times \left| 1 + \sum_{k=1}^N c'_k (\mu_n + te^{i\theta/3})^{-3k/2} + r_N(\mu_n + te^{i\theta/3}) \right|^2 dt \end{aligned}$$

where

$$|r_N(\xi)| = \mathcal{O}(|\xi|^{-3N/2}), \quad |\xi| \rightarrow +\infty,$$

since  $|\mu_n + te^{i\theta/3}| \rightarrow +\infty$  as  $n \rightarrow +\infty$  uniformly with respect to  $t \in [\varepsilon|\mu_n|, +\infty[$  :

$$\forall t \geq 0, \quad |\mu_n + te^{i\theta/3}| \geq |t \sin \theta/3| \geq \varepsilon|\mu_n| |\sin \theta/3| \rightarrow +\infty, \quad n \rightarrow +\infty.$$

Putting

$$h = h_n := |\mu_n|^{-3/2} \quad \text{and} \quad t = h^{-2/3}s = |\mu_n|s, \quad (2.14)$$

we then get

$$I_{n,\varepsilon} = \hat{I}_\varepsilon(h_n)$$

where

$$\hat{I}_\varepsilon(h) = \frac{1}{4\pi h^{1/3}} \int_\varepsilon^{+\infty} a_\theta(s, h) e^{-\frac{1}{h} \varphi_\theta(s)} ds, \quad (2.15)$$

with

$$a_\theta(s, h) \underset{h \rightarrow 0}{\sim} \frac{1}{|-1 + se^{i\theta/3}|^{1/2}} \left( 1 + \sum_{k=1}^{+\infty} \tilde{c}_k(s) h^k \right)$$

and

$$\varphi_\theta(s) = \frac{4}{3} \operatorname{Re}(-1 + se^{i\theta/3})^{3/2},$$

the functions  $\tilde{c}_k(s)$  satisfying, for all  $k \geq 1$ ,

$$|\tilde{c}_k(s)| = \mathcal{O}((1+s)^{-3k/2}).$$

Now the right-hand side of (2.15) is in the right form to apply the Laplace method as  $h \rightarrow 0$ . Let us check that the function  $\varphi_\theta$  has a unique non-degenerate critical point  $s_\theta$  in the interval  $[\varepsilon, +\infty[$ , and that  $\varphi_\theta''(s_\theta) > 0$ . We have

$$\begin{aligned} \varphi_\theta'(s) &= 2 \operatorname{Re} \left( e^{i\theta/3} (-1 + se^{i\theta/3})^{1/2} \right) \\ &= 2 |-1 + se^{i\theta/3}|^{1/2} \cos \left( \frac{\theta}{3} + \frac{1}{2} \arg(-1 + se^{i\theta/3}) \right). \end{aligned}$$

Since  $\arg(-1 + se^{i\theta/3}) \in ]\theta/3, \pi[$  and  $\theta < 3\pi/4$ ,  $\varphi_\theta'$  vanishes if and only if

$$\frac{\theta}{3} + \frac{1}{2} \arg(-1 + se^{i\theta/3}) = \frac{\pi}{2},$$

which yields

$$\tan \frac{2\theta}{3} = \frac{s_\theta \sin(\theta/3)}{1 - s_\theta \cos(\theta/3)},$$

hence

$$s_\theta = \frac{\tan(2\theta/3)}{\sin(\theta/3) + \cos(\theta/3) \tan(2\theta/3)} = \frac{\sin(2\theta/3)}{\sin \theta}, \quad (2.16)$$

and we see that  $s_\theta > 0$  for  $0 < \theta < 3\pi/4$ .

On the other hand, one can easily check that

$$\varphi_\theta''(s_\theta) = m_\theta^{-1/2} \sin \theta > 0 \quad (2.17)$$

where  $m_\theta := |-1 + s_\theta e^{i\theta/3}|$  is the constant appearing in Theorem 1.1. Finally, the principal term of  $a(s_\theta, h)$  is equal to

$$a(s_\theta, 0) = m_\theta^{-1/2} \neq 0,$$

and

$$\varphi_\theta(s_\theta) = -\frac{4}{3} m_\theta^{3/2} \sin \theta.$$

We choose  $\varepsilon < s_\theta$  and  $M > s_\theta$  fixed, so that  $\varphi_\theta(\varepsilon) > \varphi_\theta(s_\theta)$  and  $\varphi_\theta(M) > \varphi_\theta(s_\theta)$ . We write

$$\hat{I}_\varepsilon(h) = \hat{I}_{\varepsilon,M}(h) + \hat{R}_M(h), \quad (2.18)$$

with

$$\begin{cases} \hat{I}_{\varepsilon,M}(h) &= \frac{1}{4\pi h^{1/3}} \int_\varepsilon^M a_\theta(s, h) e^{-\frac{1}{h}\varphi_\theta(s)} ds \\ \hat{R}_M(h) &= \frac{1}{4\pi h^{1/3}} \int_M^{+\infty} a_\theta(s, h) e^{-\frac{1}{h}\varphi_\theta(s)} ds. \end{cases}$$

The Laplace method applies to  $\hat{I}_{\varepsilon,M}(h)$  and yields, according to the computations above,

$$\hat{I}_{\varepsilon,M}(h) \underset{h \rightarrow 0}{\sim} \frac{h^{1/6}}{2\sqrt{2\pi}m_\theta^{1/4}\sqrt{\sin\theta}} \exp\left(\frac{4}{3h}m_\theta^{3/2}\sin\theta\right) \left(1 + \sum_{k=1}^{+\infty} A_k(\theta)h^k\right) \quad (2.19)$$

for a real sequence  $(A_k(\theta))_{k \geq 1}$ .

Regarding  $\hat{R}_M(h)$ , since  $\varphi_\theta$  is non-decreasing on  $[M, +\infty[$ , we have

$$\begin{aligned} \hat{R}_M(h) &\leq \frac{K}{h^{1/3}} \int_M^{+\infty} e^{-\frac{1}{h}\varphi_\theta(s)} ds, \quad K > 0, \\ &\leq Kh^{-1/3} e^{(1-\frac{1}{h})\varphi_\theta(M)} \int_M^{+\infty} e^{-\varphi_\theta(s)} ds \\ &\leq K'h^{-1/3} e^{-\frac{1}{h}\varphi_\theta(M)}, \quad K' > 0, \end{aligned} \quad (2.20)$$

as soon as  $h < 1$ , and of course  $\varphi_\theta(M) > \varphi_\theta(s_\theta)$ .

Using (2.18), (2.19) and (2.20), we get

$$I_{n,\varepsilon} \underset{n \rightarrow +\infty}{\sim} \frac{h^{1/6}}{2\sqrt{2\pi}m_\theta^{1/4}\sqrt{\sin\theta}} \exp\left(\frac{4}{3h}m_\theta^{3/2}\sin\theta\right) \left(1 + \sum_{k=1}^{+\infty} A_k(\theta)h^k\right). \quad (2.21)$$

**Remark 2.4** In the case of the operator  $-\frac{d^2}{dx^2} + ix$ , which is studied in [3] and [14], the critical point of  $\varphi_{\pi/2}$  is reached when

$$\arg(-1 + se^{i\theta/3}) = \frac{2\pi}{3}.$$

We then get the following values:  $s_{\pi/2} = \sqrt{3}/2$ ,  $\varphi_{\pi/2}(s_{\pi/2}) = -\sqrt{2}/3$ ,  $\varphi''_{\pi/2}(s_{\pi/2}) = \sqrt{2}$  and  $a(s_{\pi/2}, 0) = \sqrt{2}$ , which yields in this case

$$I_{n,\varepsilon} \underset{n \rightarrow +\infty}{\sim} \frac{h^{1/6}}{2^{5/4}\sqrt{\pi}} e^{\frac{\sqrt{2}}{3h}} \left(1 + \sum_{k=1}^{+\infty} A_k(\pi/2)h^k\right). \quad (2.22)$$

**Step 1.b.** It remains to estimate the second term in (2.13),

$$R_{n,\varepsilon} = \int_0^{\varepsilon|\mu_n|} |Ai(\mu_n + te^{i\theta/3})|^2 dt,$$

but a rough control of this remainder term will be enough.

For  $n$  large enough, we can apply (2.7) uniformly with respect to  $t \in [0, \varepsilon]$  to get

$$\begin{aligned} R_{n,\varepsilon} &\leq C|\mu_n| \sup_{t \in [0, \varepsilon]|\mu_n|} \frac{1}{|\mu_n + te^{i\theta/3}|^{1/2}} \left| \sin \left( \frac{2}{3}(-(\mu_n + te^{i\theta/3}))^{3/2} + \frac{\pi}{4} \right) \right|^2, \quad C > 0, \\ &\leq C' \frac{\varepsilon|\mu_n|^{1/2}}{|-1 + \varepsilon e^{i\theta/3}|^{1/2}} \left| \sin \left( \frac{2}{3}(|\mu_n|(-1 + \varepsilon e^{i\theta/3}))^{3/2} + \frac{\pi}{4} \right) \right|^2, \quad C' > 0, \\ &\leq C'' \frac{\varepsilon}{\pi|-1 + \varepsilon e^{i\theta/3}|^{1/2} h_n^{1/3}} \exp \left( -\frac{1}{h_n} \varphi_\theta(\varepsilon) \right), \quad C'' > 0, \end{aligned}$$

where we used

$$\left| \sin \left( \frac{2}{3} z^{3/2} + \frac{\pi}{4} \right) \right| \leq \frac{1}{2} \exp \left( -\frac{2}{3} \operatorname{Re} z^{3/2} \right) + o(1)$$

for  $2\pi/3 < |\arg z| < \pi$ . Thus, it exists  $C = C(\theta, \varepsilon) > 0$  such that, for  $n$  large enough,

$$R_{n,\varepsilon} \leq C h^{-1/3} \exp \left( -\frac{1}{h} \varphi_\theta(\varepsilon) \right).$$

Using that, for  $\varepsilon < s_\theta$ ,

$$-\varphi_\theta(\varepsilon) < -\varphi_\theta(s_\theta) = \frac{4}{3} m_\theta^{3/2} \sin \theta,$$

(2.13) implies

$$\int_{\mathbb{R}} |u_n(t)|^2 dt = I_{n,\varepsilon} (1 + o(e^{-c/h_n}))$$

for some  $c > 0$ . The expansion (2.21) finally yields the proposition:

**Proposition 2.5** *Let  $u_n(x) = \operatorname{Ai}(\mu_n + x e^{i\theta/3})$  and  $h_n = |\mu_n|^{-3/2}$ , then*

$$\|u_n\|^2 \underset{n \rightarrow +\infty}{\sim} \frac{h_n^{1/6}}{2\sqrt{2\pi} m_\theta^{1/4}} \exp \left( \frac{4}{3 h_n} m_\theta^{3/2} \sin \theta \right) \left( 1 + \sum_{k=1}^{+\infty} A_k(\theta) h_n^k \right). \quad (2.23)$$

This concludes the analysis of the numerator of (2.3).

### Step 2: The $J_n$ term

It remains to estimate the denominator of (2.3). We first recover a real integral, using the analyticity of the Airy function and its decay at infinity in the sector  $|\arg z| < \pi/3$ :

$$\begin{aligned} e^{i\theta/3} \int_{\mathbb{R}^+} u_n^2(x) dx &= \int_0^{+\infty} e^{i\theta/3} \operatorname{Ai}^2(e^{i\theta/3} x + \mu_n) dx \\ &= \lim_{R \rightarrow +\infty} \int_{\gamma_R} \operatorname{Ai}^2(z) dz, \end{aligned}$$

where  $\gamma_R$  is the segment  $[\mu_n, \mu_n + e^{i\theta/3}R] \subset \mathbb{C}$ . The Airy function is holomorphic in  $\mathbb{C}$  and  $\gamma_R$  can be decomposed into

$$\gamma_R = [\mu_n, \mu_n + R] \vee \mathcal{C}_R$$

where  $\mathcal{C}_R$  is the arc of a circle (centered at  $\mu_n$ , of radius  $R$ )

$$\mathcal{C}_R : [0, 1] \ni t \mapsto \mu_n + Re^{it\theta/3}.$$

Hence,

$$\begin{aligned} e^{i\theta/3} \int_{\mathbb{R}^+} u_n^2(x) dx &= \lim_{R \rightarrow +\infty} \left( \int_{\mu_n}^{\mu_n+R} Ai^2(x) dx + \int_{\mathcal{C}_R} Ai^2(z) dz \right) \\ &= \int_{\mu_n}^{+\infty} Ai^2(x) dx \end{aligned}$$

since, using (2.6), the integral  $\int_{\mathcal{C}_R} Ai^2(x) dx$  tends to 0 as  $R \rightarrow +\infty$ . Thus,

$$\left| \int_{\mathbb{R}^+} u_n^2(x) dx \right| = \int_{\mu_n}^{+\infty} Ai^2(x) dx, \quad (2.24)$$

since  $Ai(x)$  is real for real  $x$ .

Now we use the formula of Lemma 2.3, which yields

$$\int_{\mu_n}^{+\infty} Ai^2(x) dx = Ai'^2(\mu_n). \quad (2.25)$$

Here we have used that (2.6) and (2.8) imply

$$xAi^2(x) - Ai'^2(x) \rightarrow 0$$

as  $x \rightarrow +\infty$ , and  $Ai(\mu_n) = 0$ .

According to the expansion (2.9) of  $Ai'$ , with the notation of (2.14), there exists a real sequence  $(d'_k)_{k \geq 0}$  such that

$$\begin{aligned} \left| \int_{\mathbb{R}^+} u_n^2(x) dx \right| &\underset{n \rightarrow +\infty}{\sim} \frac{1}{\pi} h_n^{-1/3} \left( \cos \left( \frac{2}{3h_n} + \frac{\pi}{4} \right) \sum_{k=0}^{+\infty} (-1)^k d'_{2k} h_n^{2k} \right. \\ &\quad \left. + \sin \left( \frac{2}{3h_n} + \frac{\pi}{4} \right) \sum_{k=0}^{+\infty} (-1)^k d'_{2k+1} h_n^{2k+1} \right)^2. \end{aligned} \quad (2.26)$$

The expansion (2.10) of  $|\mu_n|$  gives

$$h_n^{-1} \underset{n \rightarrow +\infty}{\sim} \frac{3\pi}{2} (n - 1/4) \left( 1 + \sum_{k=1}^{+\infty} \gamma_k (n - 1/4)^{-2k} \right), \quad (\gamma_k)_{k \geq 1} \in \mathbb{R}^{\mathbb{N}}. \quad (2.27)$$

Hence, for some sequence  $(\delta_k)_{k \geq 1}$ ,

$$\begin{aligned} \cos\left(\frac{2}{3h_n} + \frac{\pi}{4}\right) \sum_{k=0}^{+\infty} (-1)^k d'_{2k} h_n^{2k} + \sin\left(\frac{2}{3h_n} + \frac{\pi}{4}\right) \sum_{k=0}^{+\infty} (-1)^k d'_{2k+1} h_n^{2k+1} \\ \underset{n \rightarrow +\infty}{\sim} (-1)^n \left(1 + \sum_{k=1}^{+\infty} \delta_{2k} (n - 1/4)^{-2k}\right). \end{aligned}$$

From (2.26), we finally get

**Proposition 2.6** *There exists  $(B_k)_{k \geq 1}$  such that*

$$\left| \int_{\mathbb{R}^+} u_n^2(x) dx \right| \underset{n \rightarrow +\infty}{\sim} \frac{1}{\pi} h_n^{-1/3} \left(1 + \sum_{k=1}^{+\infty} B_k (n - 1/4)^{-2k}\right), \quad (2.28)$$

where  $h_n$  is defined by (2.14).

### Step 3: Conclusion

Writing each expansion in powers of  $h_n$  and  $(n - 1/4)^{-2}$  as expansions in powers of  $n^{-1}$ , we deduce from (2.3), (2.23) and (2.28) that for some  $(C_k(\theta))_{k \geq 1}$ ,

$$\begin{aligned} \kappa_n^D(1, \theta) \underset{n \rightarrow +\infty}{\sim} \frac{\sqrt{\pi}}{2\sqrt{2}m_\theta^{1/4}\sqrt{\sin \theta}} h_n^{1/2} \exp\left(\frac{4}{3h_n} m_\theta^{3/2} \sin \theta\right) \\ \times \left(1 + \sum_{k=1}^{+\infty} C_k(\theta) n^{-k}\right). \end{aligned}$$

Finally, (2.27) yields

$$\begin{aligned} h_n^{1/2} \exp\left(\frac{4}{3h_n} m_\theta^{3/2} \sin \theta\right) \underset{n \rightarrow +\infty}{\sim} \sqrt{\frac{2}{3\pi n}} \exp\left(2\pi m_\theta^{3/2} \sin \theta (n - 1/4)\right) \\ \times \left(1 + \sum_{k=1}^{+\infty} C'_k(\theta) n^{-k}\right), \end{aligned}$$

and the statement of Theorem 1.1 follows. The particular case of  $\kappa_n^D(1, \pi/2)$  follows from (2.22).

The case of the Neumann realization is similar, putting  $\tilde{h}_n = |\nu_n|^{-3/2}$  and using (2.11) instead of (2.10). Notice however that (2.25) becomes in this case

$$\int_{\nu_n}^{+\infty} Ai^2(x) dx = |\nu_n| Ai^2(\nu_n),$$

hence we use (2.7) instead of (2.9) (this was wrongly stated in [18]).

In the next section, we will perform a similar analysis for the instability indices  $\kappa_n(2k, \theta)$ ,  $k \geq 1$ . The asymptotic expansions of the Airy function will be replaced by WKB estimates on the eigenfunctions of  $\mathcal{A}(2k, \theta)$ .

### 3 Instability of even anharmonic oscillators

We would like to understand the behavior as  $n \rightarrow +\infty$  of the instability indices  $\kappa_n(2k, \theta)$ ,  $k \geq 1$ , using formule (1.6). In this purpose, we first consider the solutions  $\psi_h$  of the selfadjoint operator  $-h^2 \frac{d^2}{dx^2} + x^{2k} - 1$ . We will then explain the relation between  $\psi_h$  and the eigenfunctions of  $\mathcal{A}(2k, \theta)$  in Paragraph 3.2.

#### 3.1 WKB estimates for the eigenfunctions

We use the complex WKB method (see appendix) to understand the asymptotic behavior of the solutions  $\psi_h \in L^2(\mathbb{R})$  of equation

$$\mathcal{P}_h(2k)\psi_h = \left(-h^2 \frac{d^2}{dx^2} + x^{2k} - 1\right)\psi_h = 0, \quad (3.1)$$

where  $k \geq 1$ . The eigenfunctions of the operator  $\mathcal{A}(2k, \theta)$  are indeed related to the functions  $\psi_h$ , since the transformation  $x \mapsto |\lambda|^{-1/2k} e^{i\theta/(2k+2)} x$  maps formally  $\mathcal{A}(2k, \theta)$  into the operator  $\mathcal{P}_h(2k)$  of (3.1), with  $h = |\lambda|^{-\frac{k+1}{2k}}$  (see the details in paragraph 3.2). Consequently, we are especially concerned with the behavior of  $\psi_h(x)$  as  $x$  goes to infinity in the sector

$$\mathcal{S}_{\theta/(2k+2)} = \left\{x \in \mathbb{C} : |\arg x| \leq \frac{\theta}{2(k+1)}\right\}, \quad (3.2)$$

but also at the limit  $h \rightarrow 0$ .

Let  $x_j = e^{ij\pi/k}$ ,  $j = 0, \dots, (2k-1)$ , denote the zeroes of the function  $x \mapsto x^{2k} - 1$ , and let us consider the function  $x \mapsto \sqrt{x^{2k} - 1}$  defined on the set

$$D = \mathbb{C} \setminus \bigcup_{j=0}^{2k-1} e^{ij\pi/k} [1, +\infty[.$$

We then denote by  $S(z)$  the function

$$S : z \mapsto \int_1^z \sqrt{x^{2k} - 1} \, dx,$$

where the integral is taken along a path defined on  $D$  joining 1 and  $z$ , the integral being independent of the path.

If  $\mathcal{L}$  denotes the union of level sets (also referred as *Stokes lines*) of the function  $\operatorname{Re} S$  starting from the zeroes  $x_j = e^{ij\pi/k}$ ,  $j = 0, \dots, (2k-1)$ , then  $\mathbb{C} \setminus \mathcal{L}$  consists of  $2k+2$  unbounded connected components  $\Sigma_j$ , each domain  $\Sigma_j$  being bounded at infinity by the asymptotic directions  $\mathbb{R}^+ e^{i(2j-1)\pi/(2k+2)}$  and  $\mathbb{R}^+ e^{i(2j+1)\pi/(2k+2)}$ , along with the two domains

$$\mathcal{D}_{\pm} = \left( \mathbb{C} \setminus \bigcup_{j=0}^{2k+1} \overline{\Sigma_j} \right) \cap \{\operatorname{Im} z \gtrless 0\} \quad ;$$

see the following figures.

Let  $\Gamma = \mathcal{D}_+ \cup \mathcal{D}_- \cup \Sigma_0 \cup \Sigma_{-1} \cup \Sigma_{+1}$ , and for  $\varepsilon > 0$ ,

$$\Gamma_\varepsilon = \{z \in \Gamma : d(z, \partial\Gamma) \geq \varepsilon\}, \quad (3.3)$$

$\partial\Gamma$  being the boundary of  $\Gamma$ .

Then we have the following statement, where the asymptotics as  $|z| \rightarrow +\infty$  also hold for a potential  $x^{2k} - e^{i\mu}$ ,  $\mu \in \mathbb{R}$ , instead of  $x^{2k} - 1$ .

**Proposition 3.1**

1. For all  $\mu \in \mathbb{R}$  and  $\eta \in ]-\pi/(2k+2), \pi/(2k+2)[$ , if  $\psi_h$  solves (3.1) or

$$\left(-h^2 \frac{d^2}{dx^2} + x^{2k} - e^{i\mu}\right) \psi_h = 0, \quad (3.4)$$

with  $\psi_h \in L^2(\mathbb{R}^+ e^{i\eta})$ , then there exists  $c \in \mathbb{C}$  such that, for all  $\alpha \in ]-\pi/(2k+2), \pi/(2k+2)[$ ,

$$\psi_h(z) = \frac{c}{(z^{2k} - 1)^{1/4}} \exp\left(-\frac{1}{h} S(z)\right) (1 + o(1)), \quad z = re^{i\alpha}, \quad r \rightarrow +\infty, \quad (3.5)$$

uniformly with respect to  $h$ . Moreover,  $\psi_h$  is the unique solution satisfying (3.5) for  $\alpha \in ]-\pi/(2k+2), \pi/(2k+2)[$ , and  $\psi_h$  is holomorphic in  $\mathbb{C}$ .

2. In the case  $\mu = 0$ , there exists a sequence of functions  $u_j$  defined on  $\Gamma$ ,  $j \geq 1$ , such that

$$\psi_h(z) \underset{h \rightarrow 0}{\sim} \frac{c}{(z^{2k} - 1)^{1/4}} \left(1 + \sum_{j=1}^{+\infty} u_j(z) h^j\right) \exp\left(-\frac{1}{h} S(z)\right) \quad (3.6)$$

uniformly on  $\Gamma_\varepsilon$ , for any  $\varepsilon > 0$ .

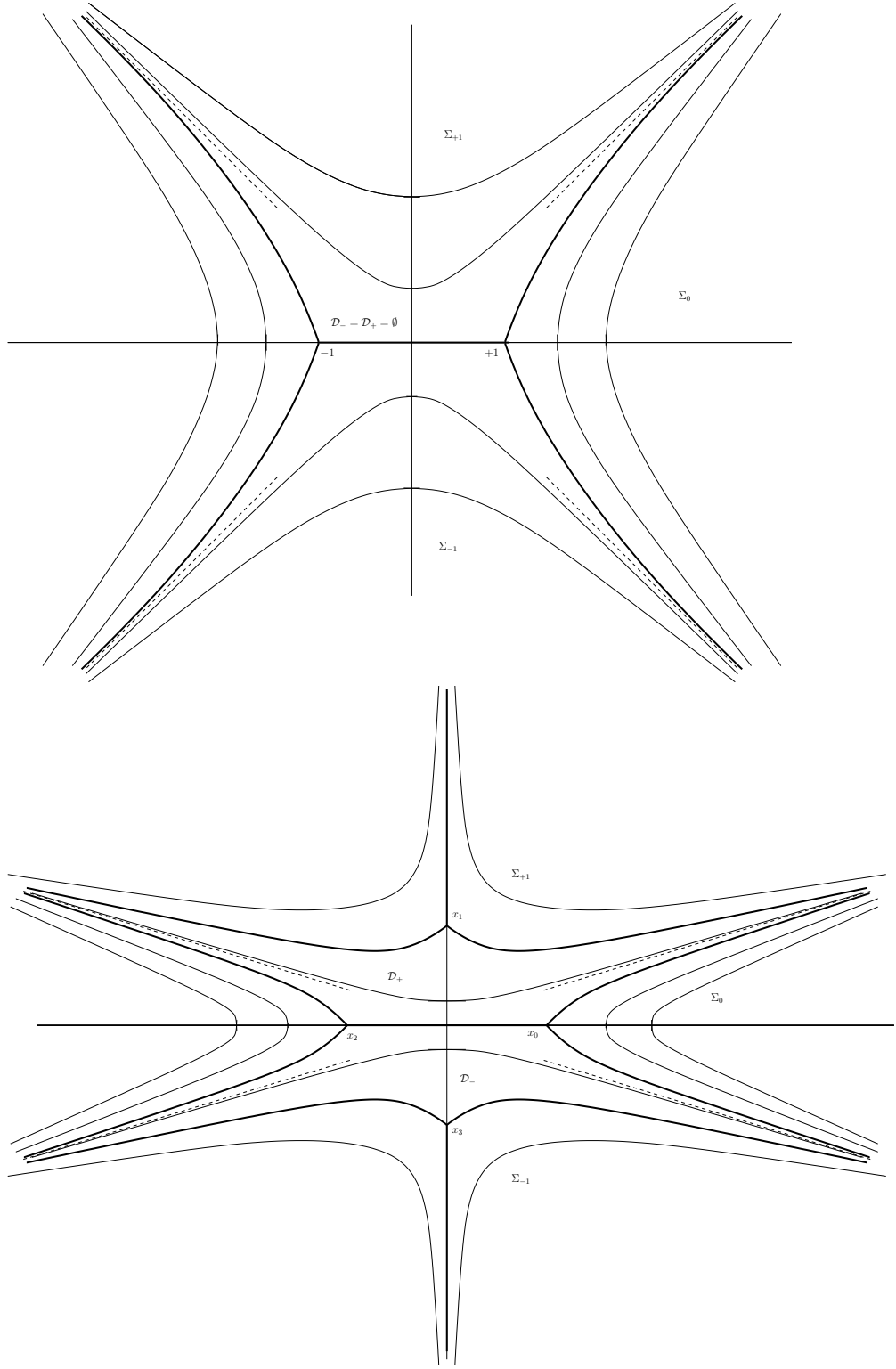


Figure 4: Stokes lines of the harmonic and quartic oscillators ( $k = 1$  and  $k = 2$ ).

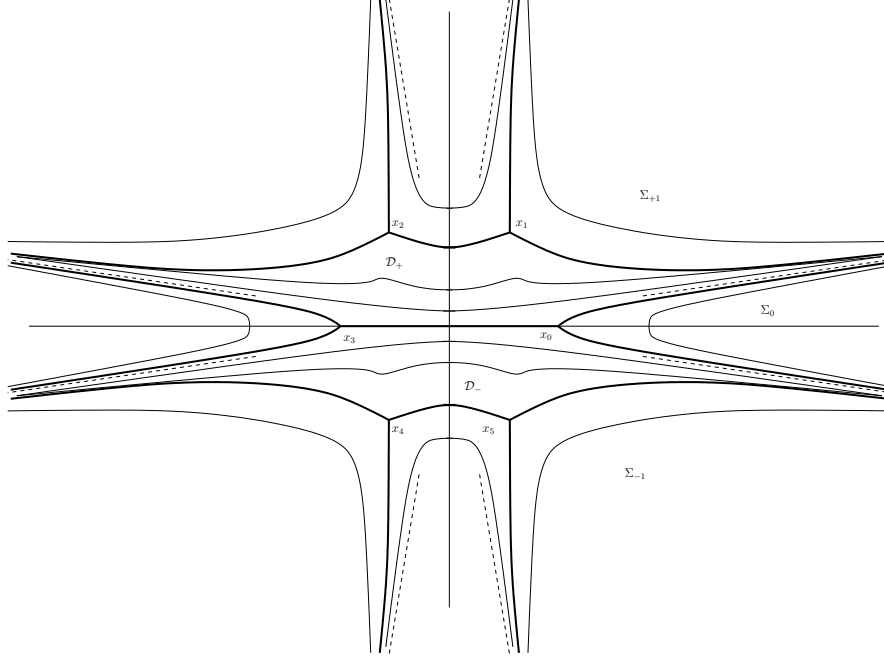


Figure 5: Stokes lines of the anharmonic oscillator  $-\frac{d^2}{dx^2} + x^6 - 1$ .

**Proof:** We apply Theorem A.1 with  $f(x) = x^{2k} - 1$ , and with base points  $a_0 = 1$  and  $a_1 = +\infty e^{i\eta}$  the point at infinity in the direction  $\arg^{-1}\{\eta\}$ . The analysis of the Stokes lines show that for all  $z \in \Sigma_{-1} \cup \Sigma_0 \cup \Sigma_1$ , one can find a canonical path joining  $a_1$  to  $z$ : it is enough to paste a ray  $[Re^{i\eta}, a_1[$ , where  $R > 0$  is such that  $\operatorname{Re} S(Re^{i\eta}) > \operatorname{Re} S(z)$ , with the path  $S^{-1}([S(z), S(Re^{i\eta})])$ . This construction holds for the solutions of (3.1) as well as for those of (3.4), the asymptotic directions of the Stokes lines being the same in both cases.

For  $\mu = 0$ , we can also find a canonical path  $\gamma(z)$  joining  $a_1$  to any point  $z \in \mathcal{D}_- \cup \mathcal{D}_+$ , so that  $\Gamma \subset \Delta(a_0, a_1)$  according to the notation of the appendix. Hence it remains to check that there exist  $k, \rho, M > 0$  such that, for any  $z \in \Gamma$ , the function  $\sigma(z)$  defined in (A.3) satisfies (A.4), and such that for all  $z \in \Gamma_\varepsilon$ , there exists a canonical path  $\gamma(z)$  joining  $a_1$  to  $z$  and satisfying (A.5).

Conditions (A.4) and (A.5) follow from the behavior of  $\sigma$  as  $|z| \rightarrow +\infty$ :

$$\sigma(z) = Cz^{-2k-2}(1 + o(1)), \quad C \in \mathbb{R} \neq 0,$$

which concludes the proof of the proposition.  $\square$

The asymptotic behavior of the solutions of (3.4) being known in the whole sector  $\arg^{-1}] - \pi/(2k+2), \pi/(2k+2)[$ , we will be able in the next paragraph to justify the transformation mentioned above to switch from  $\mathcal{A}(2k, \theta)$  to  $\mathcal{P}_h(2k)$ .

### 3.2 Another expression of the instability indices

Let  $n \geq 1$ ,  $\lambda_n$  the  $n$ -th eigenvalue of  $\mathcal{A}(2k, \theta)$ , and  $u_n$  an associated eigenfunction. The scale change  $\tilde{x} = |\lambda_n|^{-1/2k}x$  will first map the equation

$$(\mathcal{A}(2k, \theta) - \lambda_n)u_n(x) = 0$$

into

$$|\lambda_n| \left( -h_n^2 \frac{d^2}{d\tilde{x}^2} + e^{i\theta} \tilde{x}^{2k} - e^{i \arg \lambda_n} \right) u_n(h_n^{-1/(k+1)} \tilde{x}) = 0, \quad (3.7)$$

where we put

$$h_n = |\lambda_n|^{-\frac{k+1}{2k}}. \quad (3.8)$$

This will be our semi-classical parameter, observing that  $h_n \rightarrow 0$  as  $n \rightarrow +\infty$ . Then we perform the analytic dilation  $y = e^{i\theta/(2k+2)} \tilde{x}$ , and (3.7) becomes

$$|\lambda_n| e^{i\theta/(k+1)} \left( -h_n^2 \frac{d^2}{dy^2} + y^{2k} - e^{i(\arg \lambda_n - \theta/(k+1))} \right) \psi_{h_n}(y) = 0,$$

where

$$\psi_{h_n}(y) = u_n(h_n^{-1/(k+1)} e^{-i\theta/(2k+2)} y). \quad (3.9)$$

Since  $u_n \in L^2(\mathbb{R})$ , we have  $\psi_{h_n} \in L^2(e^{-i\theta/(2k+2)} \mathbb{R})$ . According to Proposition 3.1, we also have  $\psi_{h_n} \in L^2(\mathbb{R})$ , hence  $\psi_{h_n}$  is an eigenfunction of the non-negative selfadjoint operator

$$-h_n^2 \frac{d^2}{dy^2} + y^{2k},$$

associated with the eigenvalue  $e^{i(\arg \lambda_n - \theta/(k+1))}$ . Therefore, we have necessarily

$$e^{i(\arg \lambda_n - \theta/(k+1))} = 1,$$

hence we recover the operator  $\mathcal{P}_h(2k)$  defined in (3.1), and all the eigenvalues of  $\mathcal{A}(2k, \theta)$  lie on the half-line  $\arg^{-1} \{ \frac{\theta}{k+1} \}$ .

According to paragraph 3.1, we can choose to normalize the eigenfunction  $u_n$  so that the function  $\psi_h$  defined by (3.9) is characterized by (3.5) with  $c = 1$ , as  $x \rightarrow +\infty$  for instance (or equivalently for  $|x| \rightarrow \infty$  along any ray included in the sector  $\arg^{-1} ] - \pi/(2k+2), \pi/(2k+2)[$ , see paragraph 3.1).

The spectral projection associated with  $\lambda_n$  being of rank 1 according to Lemma 5 of [6], Formula (1.6) holds for  $\kappa_n(2k, \theta)$ . Using (3.9) and the scale change  $x \mapsto h_n^{\frac{1}{k+1}} x$ , we get

$$\kappa_n(2k, \theta) = \frac{\int_{\mathbb{R}} |\psi_{h_n}(e^{i\frac{\theta}{2(k+1)}} y)|^2 dy}{\left| \int_{\mathbb{R}} \psi_{h_n}^2(e^{i\frac{\theta}{2(k+1)}} y) dy \right|}.$$

Now again we write the denominator as a real integral, as in Section 2.3:

$$e^{i\frac{\theta}{2(k+1)}} \int_{-\infty}^{+\infty} \psi_{h_n}^2(e^{i\frac{\theta}{2(k+1)}} y) dy = \lim_{R \rightarrow +\infty} \int_{\gamma_R} \psi_{h_n}^2(z) dz$$

where  $\gamma_R$  is the segment  $[-Re^{i\frac{\theta}{2(k+1)}}, Re^{i\frac{\theta}{2(k+1)}}] \subset \mathbb{C}$ . We write

$$\gamma_R = [-R, +R] \vee \mathcal{C}_R \vee (-\mathcal{C}_R),$$

where  $\mathcal{C}_R(t) = Re^{it\frac{\theta}{2(k+1)}}$ ,  $t \in [0, 1]$ , and the exponential decay of  $\psi_{h_n}$  as  $|x| \rightarrow +\infty$  in the sector (3.2) yields

$$\lim_{R \rightarrow +\infty} \int_{\mathcal{C}_R} \psi_{h_n}^2(z) dz = 0 \quad (3.10)$$

(see Lemma 3.1 and notice that  $S(z) \rightarrow +\infty$  as  $|z| \rightarrow +\infty$  in the whole sector (3.2)).

Similarly,

$$\lim_{R \rightarrow +\infty} \int_{-\mathcal{C}_R} \psi_{h_n}^2(z) dz = 0.$$

Thus, we have

$$\kappa_n(2k, \theta) = \frac{\int_{\mathbb{R}} |\psi_{h_n}(e^{i\frac{\theta}{2(k+1)}} x)|^2 dx}{\int_{\mathbb{R}} \psi_{h_n}^2(x) dx}. \quad (3.11)$$

Finally, the potential in  $\mathcal{A}(2k, \theta)$  is even, hence the solutions  $\psi_{h_n}$  are even or odd, and we can replace the integrals in (3.11) by integrals over  $\mathbb{R}^+$ .

We summarize all the previous informations in the following proposition:

**Proposition 3.2** *For all  $n \geq 1$ , the  $n$ -th eigenvalue  $\lambda_n$  of the operator  $\mathcal{A}(2k, \theta)$  writes*

$$\lambda_n = r_n e^{i\theta/(k+1)},$$

where  $r_n > 0$  is the  $n$ -th eigenvalue of the selfadjoint anharmonic oscillator

$$-\frac{d^2}{dx^2} + x^{2k}.$$

The following Weyl formula holds: there exists a real sequence  $(s_j)_{j \geq 1}$  such that

$$r_n \underset{n \rightarrow +\infty}{\sim} \left( \frac{(k+1)\sqrt{\pi}\Gamma(\frac{k+1}{2k})}{\Gamma(\frac{1}{2k})} (n+1/2) \right)^{\frac{2k}{k+1}} \left( 1 + \sum_{j=1}^{+\infty} s_j (n+1/2)^{-2j} \right). \quad (3.12)$$

There exists a unique eigenfunction  $u_n$  of  $\mathcal{A}(2k, \theta)$  associated with  $\lambda_n$  such that

$$u_n(x) = \psi_{h_n}(h_n^{1/(k+1)} e^{i\theta/(2k+2)} x),$$

where  $h_n$  is defined by (3.8) and  $\psi_h$  is the unique solution of equation (3.1) satisfying (3.5) with  $c = 1$  as  $x \rightarrow +\infty$ . Lastly, we have

$$\kappa_n(2k, \theta) = \frac{\int_{\mathbb{R}^+} |\psi_{h_n}(e^{i\theta/(2k+2)} x)|^2 dx}{\int_{\mathbb{R}^+} \psi_{h_n}(x)^2 dx}. \quad (3.13)$$

The expansion (3.12) is proven in [16], Theorem (2-1), for operators of the form  $-\frac{d^{2\ell}}{dx^{2\ell}} + P(x)$ , where  $P$  is a real polynomial of even degree.

In the next paragraph, we use the asymptotic expansions given in Lemma 3.1, with  $h = h_n \rightarrow 0$ , to get an asymptotic expansion for the numerator of (3.13).

### 3.3 Estimates on the norm of the eigenfunctions

We assume without loss of generality that  $\theta > 0$  (if  $\theta < 0$ , replace  $\theta$  by  $|\theta|$ ). For a fixed  $\delta > 0$  (which will be determined later in this paragraph), the half-line  $[\delta, +\infty[e^{i\theta/(2k+2)}$  belongs to the set  $\Gamma_\varepsilon$  defined in (3.3),  $\varepsilon < \delta$ , hence using (3.6), we have

$$\psi_h(x) \underset{h \rightarrow 0}{\sim} \frac{1}{(x^{2k} - 1)^{1/4}} \left( 1 + \sum_{j=1}^{+\infty} u_j(x) h^j \right) \exp \left( -\frac{1}{h} S(x) \right) \quad (3.14)$$

uniformly for  $x \in [\delta, +\infty[e^{i\theta/(2k+2)}$ .

Then we write

$$\int_0^{+\infty} |\psi_h(e^{i\frac{\theta}{2(k+1)}} x)|^2 dx = I_\delta(h) + R_\delta(h) \quad (3.15)$$

where

$$I_\delta(h) = \int_\delta^{+\infty} |\psi_h(e^{i\frac{\theta}{2(k+1)}} x)|^2 dx, \quad R_\delta(h) = \int_0^\delta |\psi_h(e^{i\frac{\theta}{2(k+1)}} x)|^2 dx,$$

and we first estimate  $I_\delta(h)$ . The expansion being uniform with respect to  $x$ , we can take the integral over  $[\delta, +\infty[$  in (3.14). Thus there exists a sequence  $(v_j)_{j \geq 1}$  of functions such that

$$I_\delta(h) = \int_\delta^{+\infty} a_\theta(x, h) e^{\frac{2}{h} \varphi_{\theta, k}(x)} dx \quad (3.16)$$

where

$$a_\theta(x, h) \underset{h \rightarrow 0}{\sim} \frac{1}{|x^{2k} e^{i\frac{k\theta}{k+1}} - 1|^{1/2}} \left( 1 + \sum_{j=1}^{+\infty} v_j(x) h^j \right)$$

and

$$\varphi_{\theta, k}(x) = -\operatorname{Re} \int_0^{x e^{i\frac{\theta}{2(k+1)}}} (t^{2k} - 1)^{1/2} dt.$$

Now we apply the Laplace method to  $I_\delta(h)$ , checking first that the phase  $\varphi_{\theta, k}$  has a unique non-degenerate critical point in  $[\delta, +\infty[$ .

We have

$$\varphi_{\theta, k}(x) = - \int_0^x |s^{2k} e^{i\frac{k\theta}{k+1}} - 1|^{1/2} \cos \left( \frac{1}{2} \arg(s^{2k} e^{i\frac{k\theta}{k+1}} - 1) + \frac{\theta}{2(k+1)} \right) ds,$$

hence

$$\varphi'_{\theta, k}(x) = -|x^{2k} e^{i\frac{k\theta}{k+1}} - 1|^{1/2} \cos \left( \frac{1}{2} \arg(x^{2k} e^{i\frac{k\theta}{k+1}} - 1) + \frac{\theta}{2(k+1)} \right).$$

Since  $\arg(x^{2k} e^{i\frac{k\theta}{k+1}} - 1) \in ]k\theta/(k+1), \pi]$ , we have

$$\frac{1}{2} \arg(x^{2k} e^{i\frac{k\theta}{k+1}} - 1) + \frac{\theta}{2(k+1)} \in \left] \frac{\theta}{2}, \frac{\pi}{2} + \frac{\theta}{2(k+1)} \right],$$

and  $\varphi'_{\theta,k}(x)$  vanishes if and only if

$$\frac{1}{2} \arg(x^{2k} e^{i \frac{k\theta}{k+1}} - 1) + \frac{\theta}{2(k+1)} = \frac{\pi}{2}, \quad (3.17)$$

that is

$$\frac{x^{2k} \sin(k\theta/(k+1))}{x^{2k} \cos(k\theta/(k+1)) - 1} = -\tan \frac{\theta}{k+1},$$

which gives a unique critical point  $x_{\theta,k}$  for  $\varphi_{\theta,k}$ ,

$$x_{\theta,k} = \left( \frac{\tan(\theta/(k+1))}{\sin(k\theta/(k+1)) + \cos(k\theta/(k+1)) \tan(\theta/(k+1))} \right)^{\frac{1}{2k}}. \quad (3.18)$$

One can easily check that this is a non-degenerate maximum.

Of course  $\varphi_{\theta,k}(\delta) < \varphi_{\theta,k}(x_{\theta,k})$  if  $\delta < x_{\theta,k}$ , and for  $M > x_{\theta,k}$ , the integral  $\int_M^{+\infty} a_{\theta}(x, h) e^{\frac{2}{h} \varphi_{\theta,k}(x, h)} dx$  can be estimated as in (2.20).

Thus, the Laplace method applies to (3.16), and there exists a sequence  $(r_j(2k, \theta))_{j \geq 1}$  such that

$$I_{\delta}(h) \underset{h \rightarrow 0}{\sim} \frac{\sqrt{2\pi}}{|(x_{\theta,k}^{2k} e^{ik\theta/(k+1)} - 1) \varphi''_{\theta,k}(x_{\theta,k})|^{1/2}} e^{\frac{2}{h} \varphi_{\theta,k}(x_{\theta,k})} h^{1/2} \left( 1 + \sum_{j=1}^{+\infty} r_j(2k, \theta) h^j \right). \quad (3.19)$$

The following lemma gives a rough estimate on the remainder term  $R_{\delta}(h)$  in (3.15), provided that  $\delta$  is chosen small enough.

**Lemma 3.3** *There exist  $\delta > 0$  and  $c \in ]0, 2\varphi_{\theta,k}(x_{\theta,k})[$  such that*

$$R_{\delta}(h) = \mathcal{O}(e^{c/h}). \quad (3.20)$$

**Proof:** The function  $\psi_h$  being holomorphic in  $\mathbb{C}$  (see Proposition 3.1), we can apply the maximum principle in a complex neighborhood of  $[-1, 1]$ .

Let  $a > 0$ , and  $\tilde{\delta} > 0$  small enough so that

$$\forall x \in [-1-a, 1+a] + i[-\tilde{\delta}, \tilde{\delta}], \quad \operatorname{Re} S(x) > \operatorname{Re} S(x_{\theta,k} e^{i \frac{\theta}{2(k+1)}}). \quad (3.21)$$

Let  $\Omega \subset \mathbb{C}$  be the open ellipse centered at 0, with axes  $[-1-a, 1+a]$  and  $[-i\tilde{\delta}, i\tilde{\delta}]$ .

Let  $\gamma_d = \partial\Omega \cap \Gamma_{\varepsilon}$ , where  $\Gamma_{\varepsilon}$  is the domain defined in (3.3), and  $\gamma_g = \partial\Omega \setminus \gamma_d$ .

According to Proposition 3.1, we have

$$\forall x \in \gamma_d, \quad \psi_h(x) \underset{h \rightarrow 0}{\sim} \frac{1}{(x^{2k} - 1)^{1/4}} \left( 1 + \sum_{j=1}^{+\infty} u_j(x) h^j \right) \exp \left( -\frac{1}{h} S(x) \right).$$

Hence, using (3.21),

$$\forall x \in \gamma_d, \quad |\psi_h(x)|^2 = \mathcal{O}(e^{c_d/h}) \quad \text{where} \quad c_d < 2\varphi_{\theta,k}(x_{\theta,k}).$$

On the other hand, the symmetry of eigenfunctions yields

$$\forall x \in \gamma_g, \quad |\psi_h(x)|^2 = \mathcal{O}(e^{c_g/h}) \quad \text{where} \quad c_g < 2\varphi_{\theta,k}(x_{\theta,k}),$$

and the maximum principle applied to  $\psi_h$  in  $\Omega$  gives

$$\sup_{x \in \Omega} |\psi_h(x)|^2 \leq \sup_{x \in \partial\Omega} |\psi_h(x)|^2 = \mathcal{O}(e^{c/h}), \quad c < 2\varphi_{\theta,k}(x_{\theta,k}),$$

which yields (3.20) as soon as  $\delta$  is small enough to ensure that  $\bar{D}(0, \delta) \subset \Omega$ .  $\square$

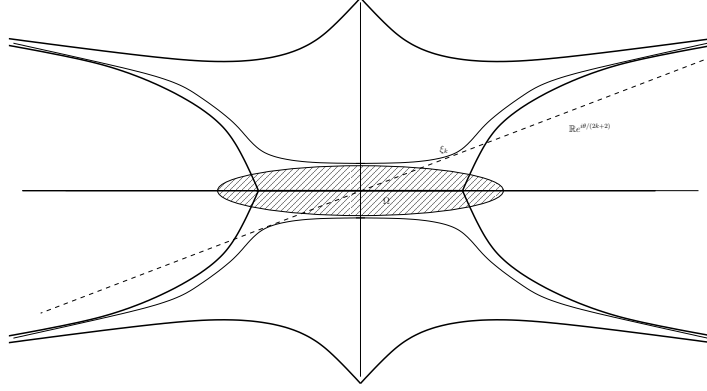


Figure 6: The open set  $\Omega$ .

Thus (3.15) and (3.19) lead to the

**Proposition 3.4** *There exists a sequence  $(r_j(2k, \theta))_{j \geq 1}$  such that*

$$\int_{\mathbb{R}} |\psi_h(e^{i\frac{\theta}{2(k+1)}} x)|^2 dx \underset{h \rightarrow 0}{\sim} C_k(\theta) h^{1/2} e^{\frac{c_k(\theta)}{h}} \left( 1 + \sum_{j=1}^{+\infty} r_j(2k, \theta) h^j \right), \quad (3.22)$$

where

$$C_k(\theta) = 2 \frac{\sqrt{2\pi}}{|(x_{\theta,k}^{2k} e^{ik\theta/(k+1)} - 1) \varphi''_{\theta,k}(x_{\theta,k})|^{1/2}}$$

and

$$c_k(\theta) = 2\varphi_{\theta,k}(x_{\theta,k}).$$

In the following paragraph, we get an asymptotic expansion for the denominator of (3.13).

### 3.4 Solutions of the selfadjoint oscillator on the real axis

The WKB expansions established in Proposition 3.1 are no longer available on the real axis, because (3.6) does not hold in the neighborhood of the Stokes line  $[-1, 1]$ . In order to estimate the denominator  $\int_0^{+\infty} \psi_h^2(x) dx$  of (3.13), we will use another representation of the function  $\psi_h$ , like in [13] (Exercise 12.3).

**Lemma 3.5** *There exists  $\delta > 0$  such that, as  $n \rightarrow +\infty$ ,*

$$\|\psi_{h_n}\|_{L^2(\mathbb{R})}^2 = \|u(\cdot, h_n)\|_{L^2(\mathbb{R})}^2 + \mathcal{O}(h_n^\infty), \quad (3.23)$$

where the function  $u(\cdot, h)$  satisfies:

- $u(\cdot, h_n)$  is an asymptotic solution of (3.1), namely

$$\mathcal{P}_{h_n}(2k)u(\cdot, h_n) = \mathcal{O}(h_n^\infty).$$

- There exist  $\mathcal{C}^\infty$  functions  $\psi_\pm$  defined in a neighborhood of 0 such that

$$(\partial_\xi \psi_\pm(\xi))^{2k} + \xi^2 - 1 = 0, \quad \psi_\pm(0) = 0, \quad \begin{cases} \partial_\xi \psi_-(0) &= 1 \\ \partial_\xi \psi_+(0) &= -1 \end{cases}, \quad (3.24)$$

and functions

$$b_\pm(\xi, h) \sim \sum_{j=0}^{+\infty} b_\pm^j(\xi) h^j, \quad b(0) \neq 0,$$

such that

$$\forall x \in [\pm 1 - 2\delta, \pm 1 + 2\delta], \quad u(x, h) = \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{\frac{i}{h}(x\xi + \psi_\pm(\xi))} b_\pm(\xi, h) d\xi. \quad (3.25)$$

- There exists  $C > 0$  and a sequence of real functions  $(a_j(x))_{j \geq 0}$ , where

$$a_0(x) = \frac{C}{(1 - x^{2k})^{1/4}}, \quad (3.26)$$

such that, for all  $x \in [-1 + \delta, 1 - \delta]$ ,

$$\begin{aligned} u(x, h) &= \cos\left(\frac{1}{h}\varphi(x) - \frac{\pi}{4}\right) \sum_{p=0}^{+\infty} (-1)^p a_{2p}(x) h^{2p} \\ &\quad + \sin\left(\frac{1}{h}\varphi(x) - \frac{\pi}{4}\right) \sum_{p=0}^{+\infty} (-1)^{p+1} a_{2p+1}(x) h^{2p+1} + \mathcal{O}(h^\infty), \end{aligned} \quad (3.27)$$

where

$$\varphi(x) = \int_{-1}^x \sqrt{1 - t^{2k}} dt.$$

- For all  $x \leq -1 - \delta$  and all  $x \geq 1 + \delta$ , we have

$$u(x, h) = \mathcal{O}(h^\infty).$$

**Proof:** The construction of  $u(\cdot, h)$  is suggested in Exercise 12.3 of [13] in the general case of a simple-well potential. By construction,

$$\mathcal{P}_h(2k)u(\cdot, h) = \mathcal{O}(h^\infty),$$

and the spectral theorem implies that there exists  $\alpha(h)$  such that

$$\|\psi_h - \alpha(h)u(\cdot, h)\|_{L^2(\mathbb{R})} = \mathcal{O}(h^\infty). \quad (3.28)$$

To show (3.23), it is then enough to check that  $\alpha(h) = 1$ , which can be made by connecting the two solutions  $\psi_h$  and  $u(\cdot, h)$ , comparing them both to a third solution expressed in terms of Airy function (see [21], Chapter 13, paragraph 7). Let us recall that  $\psi_h$  is the solution of  $\mathcal{P}_h(2k)\psi_h = 0$  characterized by its decay at infinity in the sector  $\arg^{-1}] - \pi/(2k+2), \pi/(2k+2)[$ . On the other hand,  $u(\cdot, h)$  is an asymptotic solution such that, for  $x \in [-1 + \delta, 1 - \delta]$ ,

$$u(x, h) = \frac{C}{(1 - x^{2k})^{1/4}} \cos\left(\frac{\pi}{4} - \frac{1}{h}\varphi(x)\right) + \mathcal{O}(h). \quad (3.29)$$

Another solution  $w_h$  is given by the following expression in the neighborhood, say, of the zero  $+1$  (see [21]):

$$w_h(x) = h^{1/2} \left( \frac{\zeta(x)}{1 - x^{2k}} \right)^{1/4} \left( Ai\left(\frac{\zeta(x)}{h^{2/3}}\right) + \mathcal{O}(h) \right), \quad (3.30)$$

where

$$\zeta(x) = \left( \frac{3}{2}S(x) \right)^{2/3}.$$

Expression (3.30) actually holds for all  $x \in [-1 + \delta, +\infty[$ , and comparing it to (3.29) as  $h \rightarrow 0$ , we get

$$w_h(x) = \frac{h^{1/2}}{C\sqrt{\pi}}(1 + \mathcal{O}(h))u(x, h).$$

On the other hand, comparing (3.30) with (3.5) as  $x \rightarrow +\infty$ ,

$$w_h(x) = \frac{h^{1/2}}{2\sqrt{\pi}}(1 + \mathcal{O}(h))\psi_h(x).$$

Thus, comparing the principal parts,

$$\psi_h(x) = \frac{2}{C}u(x, h) + \mathcal{O}(h^\infty),$$

that is the coefficient  $\alpha(h)$  in (3.28) is independent of  $h$ . Up to renormalization of the function  $b(x, h)$  defining  $u(\cdot, h)$ , we can assume that  $\alpha(h) = 1$  in (3.26), and

(3.23) follows.  $\square$

This lemma enables us to compute the norm of  $\psi_h$  on the real axis, using the expressions of  $u(\cdot, h)$ . Since expressions (3.25) and (3.27) differ from a  $\mathcal{O}(h^\infty)$  term on the intersection of their domain of validity, we can write

$$u = u_- \chi_- + u_0 \chi_0 + u_+ \chi_+ + \mathcal{O}(h^\infty),$$

where  $(\chi_-, \chi_0, \chi_+)$  is partition of unity and  $u_\pm, u_0$  are respectively the functions on the right-hand sides of (3.25) and (3.27). Thus,

$$\begin{aligned} \|u(\cdot, h)\|^2 &= \int_{\mathbb{R}} |u_- \chi_-|^2 dx + \int_{\mathbb{R}} |u_0 \chi_0|^2 dx + \int_{\mathbb{R}} |u_+ \chi_+|^2 dx \\ &\quad + 2 \int_{\mathbb{R}} |u_0|^2 \chi_0 \chi_- dx + 2 \int_{\mathbb{R}} |u_0|^2 \chi_0 \chi_+ dx + \mathcal{O}(h^\infty) \\ &=: I_-(h) + I_0(h) + I_+(h) + I_{0,-}(h) + I_{0,+}(h) + \mathcal{O}(h^\infty). \end{aligned} \quad (3.31)$$

In order to compute  $I_-(h)$ , we use (3.25) to write the norm as a triple integral as in [9]:

$$\begin{aligned} I_-(h) &= \frac{1}{2\pi h} \int_{\mathbb{R}} u_-(x) \bar{u}_-(x) \chi_-^2(x) dx \\ &= \frac{1}{2\pi h} \iiint_{[-1+\delta]_x \times \mathbb{R}_\xi \times \mathbb{R}_{\tilde{\xi}}} e^{\frac{i}{h}(x(\xi-\xi')+\psi(\xi)-\psi(\xi'))} \\ &\quad \times b(\xi, h) \bar{b}(\xi', h) \chi_-^2(x) dx d\xi d\xi', \end{aligned}$$

hence after the change of variables (dependent on  $\xi$ )

$$\begin{cases} \tilde{\xi} &= \xi' - \xi \\ \tilde{x} &= x + \partial_\xi \psi(\xi), \end{cases}$$

$$I_-(h) = \frac{1}{2\pi h} \int_{\mathbb{R}_\xi} \left( \iint_{[-1+\delta]_{\tilde{x}} \times \mathbb{R}_{\tilde{\xi}}} e^{\frac{i}{h}\Phi_\xi(\tilde{x}, \tilde{\xi})} A(\tilde{x}, \tilde{\xi}, \xi, h) d\tilde{x} d\tilde{\xi} \right) d\xi, \quad (3.32)$$

where

$$\Phi_\xi(\tilde{x}, \tilde{\xi}) = -\tilde{x}\tilde{\xi} + \partial_\xi \psi(\xi)\tilde{\xi} + \psi(\xi) - \psi(\tilde{\xi} + \xi)$$

and

$$A(\tilde{x}, \tilde{\xi}, \xi, h) = b(\xi, h) \bar{b}(\tilde{\xi} + \xi, h) \chi_-^2(\tilde{x} - \partial_\xi \psi(\xi)).$$

We now apply the stationary phase method to the integral with respect to  $(\tilde{x}, \tilde{\xi})$ . First we have

$$\partial_{\tilde{x}} \Phi_\xi(\tilde{x}, \tilde{\xi}) = -\tilde{\xi} \quad \text{and} \quad \partial_{\tilde{\xi}} \Phi_\xi(\tilde{x}, \tilde{\xi}) = -\tilde{x} + \partial_\xi \psi(\xi) - \partial_{\tilde{\xi}} \psi(\tilde{\xi} + \xi),$$

hence  $\Phi_\xi$  has a unique critical point at  $(\tilde{x}, \tilde{\xi}) = (0, 0)$ .

Moreover, for all  $\xi \in \mathbb{R}$ ,

$$\text{Hess } \Phi_\xi(0, 0) = \begin{pmatrix} 0 & -1 \\ -1 & -\partial_\xi^2 \psi(\xi) \end{pmatrix}$$

hence  $(0, 0)$  is a non-degenerate critical point, and the stationary phase method applies, with  $\Phi_\xi(0, 0) = 0$ .

Thus, it exists a sequence  $(m_j(\xi))_{j \geq 1}$  of functions such that

$$I_-(h) \underset{h \rightarrow 0}{\sim} \int_{\mathbb{R}} |b_0(\xi)|^2 |\chi_-(-\partial_\xi \psi(\xi))|^2 \left( 1 + \sum_{j=1}^{+\infty} m_j(\xi) h^j \right) d\xi,$$

that is

$$I_-(h) \underset{h \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} c_-^j h^j, \quad c_-^0 \neq 0. \quad (3.33)$$

Similarly, we show that

$$I_+(h) \underset{h \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} c_+^j h^j, \quad c_+^0 \neq 0. \quad (3.34)$$

We now handle the integral  $I_0(h)$ . According to (3.27), there exist real sequences  $(a'_j(x))_{j \geq 0}$ ,  $(a''_j(x))_{j \geq 1}$  and  $(\tilde{a}_j(x))_{j \geq 1}$ , where

$$a'_0(x) = \frac{C^2}{\sqrt{1 - x^{2k}}},$$

such that

$$\begin{aligned} |u_0(x, h)|^2 &\underset{h \rightarrow 0}{\sim} \cos^2 \left( \frac{1}{h} \varphi(x) - \frac{\pi}{4} \right) \sum_{j=0}^{+\infty} a'_j(x) h^{2j} \\ &\quad + \sin^2 \left( \frac{1}{h} \varphi(x) - \frac{\pi}{4} \right) \sum_{j=1}^{+\infty} a''_j(x) h^{2j} \\ &\quad + \cos \left( \frac{2}{h} \varphi(x) \right) \sum_{j=0}^{+\infty} \tilde{a}_j(x) h^{2j+1}. \end{aligned} \quad (3.35)$$

Let us integrate the first term of the right-hand side, which contains the principal part of the expansion of  $I_0(h)$ .

$$\begin{aligned} &\int_{\mathbb{R}} \chi_0^2(x) \cos^2 \left( \frac{1}{h} \varphi(x) - \frac{\pi}{4} \right) \sum_{j=0}^{+\infty} a'_j(x) h^{2j} dx \\ &\underset{h \rightarrow 0}{\sim} \frac{1}{2} \int_{\mathbb{R}} \chi_0^2(x) \left( 1 + \sin \frac{2}{h} \varphi(x) \right) \sum_{j=0}^{+\infty} a'_j(x) h^{2j} dx \\ &\underset{h \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} \left( \int_{\mathbb{R}} \frac{a'_j \chi_0^2}{2} dx \right) h^{2j} + R(h), \end{aligned}$$

where

$$R(h) = \frac{1}{2} \int_{\mathbb{R}} \chi_0^2(x) a'(x, h) \sin \frac{2}{h} \varphi(x) dx, \quad a'(x, h) \underset{h \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} a'_j(x) h^{2j}.$$

Successive integrations by parts (principle of non-stationary phase) yield

$$R(h) = \mathcal{O}(h^\infty),$$

hence

$$\int_{\mathbb{R}} \chi_0^2(x) \cos^2 \left( \frac{1}{h} \varphi(x) - \frac{\pi}{4} \right) a'(x, h) dx \underset{h \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} n_j h^{2j}$$

for some real sequence  $(n_j)_{j \geq 0}$ .

Performing the same computations on the two other terms in (3.35) finally yields, for some sequence  $(c_0^j)_{j \geq 0}$ ,

$$I_0(h) \underset{h \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} c_0^j h^j, \quad c_0^0 = \frac{1}{2} \int_{\mathbb{R}} \frac{C^2 \chi_0^2(x)}{\sqrt{1-x^2k}} dx. \quad (3.36)$$

Lastly, this method gives the same kind of asymptotic expansions for integrals  $I_{0,\pm}(h)$  in (3.31):

$$I_{0,\pm}(h) \underset{h \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} c_{0,\pm}^j h^j, \quad c_{0,\pm}^0 = \int_{\mathbb{R}} \frac{C^2 \chi_0(x) \chi_{\pm}(x)}{\sqrt{1-x^2k}} dx. \quad (3.37)$$

Gathering (3.33), (3.34), (3.36) and (3.37), according to (3.31), there exists  $(c_j)_{j \geq 0}$ ,  $c_0 \neq 0$ , such that

$$\|u(\cdot, h)\|^2 \underset{h \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} c_j h^j. \quad (3.38)$$

Thus, using (3.23), we have proved that

$$\int_{-\infty}^{+\infty} |\psi_h(x)|^2 dx \underset{h \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} c_j h^j, \quad c_0 \neq 0. \quad (3.39)$$

### 3.5 Conclusion: Proof of Theorem 1.2

Expressions (3.11), (3.22) and (3.39) yield the following statement:

**Theorem 3.6** *Let  $k \in \mathbb{N}^*$  and  $\theta$  satisfying  $0 < |\theta| < (k+1)\pi/2k$ . If  $\kappa_n(2k, \theta)$  denotes the norm of the  $n$ -th spectral projection of operator*

$$\mathcal{A}(2k, \theta) = -\frac{d^2}{dx^2} + e^{i\theta} x^{2k}$$

and  $\lambda_n$  its  $n$ -th eigenvalue, then there exists a sequence  $(C'_j(2k, \theta))_{j \geq 0}$  such that

$$\kappa_n(2k, \theta) \underset{n \rightarrow +\infty}{\sim} h_n^{1/2} e^{c'_k(\theta)/h_n} \sum_{j=0}^{+\infty} C'_j(2k, \theta) h_n^j, \quad C'_0(2k, \theta) > 0, \quad (3.40)$$

where  $h_n = |\lambda_n|^{-\frac{k+1}{2k}}$  and

$$c'_k(\theta) = 2\varphi_{\theta,k}(x_{\theta,k}) \quad (3.41)$$

where

$$x_{\theta,k} = \left( \frac{\tan(\theta/(k+1))}{\sin(k\theta/(k+1)) + \cos(k\theta/(k+1)) \tan(\theta/(k+1))} \right)^{\frac{1}{2k}}, \quad (3.42)$$

$$\varphi_{\theta,k}(\xi) = \operatorname{Im} \int_0^{\xi e^{i\frac{\theta}{2(k+1)}}} (1-t^{2k})^{1/2} dt. \quad (3.43)$$

Finally, we get Theorem 3.13 by replacing the asymptotic expansion (3.40) by an expansion in powers of  $n^{-1}$ , using the following quantization rule:

**Lemma 3.7** *For all  $k \geq 1$ , there exists a sequence  $(s_k^j)_{j \geq 1}$  such that*

$$\frac{1}{h_n} \underset{n \rightarrow +\infty}{\sim} \left( \frac{(k+1)\sqrt{\pi}\Gamma(\frac{k+1}{2k})}{\Gamma(\frac{1}{2k})} (n+1/2) \right) \left( 1 + \sum_{j=1}^{+\infty} s_k^j (n+1/2)^{-2j} \right).$$

This formula follows from (3.8) and (3.12).

In the last section, we prove Theorem 1.4 and Corollary 1.5.

## 4 Completeness and semigroups

### 4.1 Completeness of eigenfuctions

In this paragraph we prove Theorem 1.4. First of all, let us recall that, if  $\mathcal{H}$  is an Hilbert space and  $p \geq 1$ , the *Schatten class*  $C^p(\mathcal{H})$  denotes the set of compact operators  $\mathcal{A}$  such that

$$\|\mathcal{A}\|_p := \left( \sum_{n=1}^{+\infty} \mu_n(\mathcal{A})^p \right)^{1/p} < +\infty, \quad (4.1)$$

where  $(\mu_n(\mathcal{A}))_{n \geq 1}$  are the eigenvalues of  $(\mathcal{A}^* \mathcal{A})^{1/2}$ , repeated according to their multiplicity (see [10]). The space  $C^p(\mathcal{H})$ ,  $p \geq 1$ , is a Banach space. One can also define the space  $C^p(\mathcal{H})$  for  $p \in ]0, 1[$ , but the quantity (4.1) does not define a norm anymore, and  $C^p(\mathcal{H})$  is not a Banach space.

We already know that the resolvent  $\mathcal{A}(2k, \theta)^{-1}$  is compact for any  $k \geq 1$  and  $\theta$  satisfying (1.5). We now prove like in [22] that it actually belongs to a Schatten class:

**Lemma 4.1** For any  $\varepsilon > 0$ ,  $|\theta| < \frac{(k+1)\pi}{2k}$  and  $k \geq 1$ , we have

$$(\mathcal{A}(2k, \theta))^{-1} \in C^{\frac{k+1}{2k} + \varepsilon}(L^2(\mathbb{R})).$$

**Proof:** Let us show that, for all  $\varepsilon > 0$ , the series  $\sum \mu_n^{\frac{k+1}{2k} + \varepsilon}$  is convergent, where  $(\mu_n)_{n \geq 1}$  are the eigenvalues of

$$([\mathcal{A}(2k, \theta))^{-1}]^* (\mathcal{A}(2k, \theta))^{-1})^{1/2} = ([\mathcal{A}(2k, \theta)(\mathcal{A}(2k, \theta))^*]^{-1})^{1/2}.$$

If  $(\nu_n)_{n \geq 1}$  denote the eigenvalues of  $\mathcal{A}(2k, \theta)(\mathcal{A}(2k, \theta))^*$ , then we have to check that

$$\sum_{n=1}^{+\infty} \nu_n^{-p/2} < +\infty$$

as soon as  $p > \frac{k+1}{2k}$ .

$\mathcal{A}(2k, \theta)(\mathcal{A}(2k, \theta))^*$  is a selfadjoint operator, and if  $p(x, \xi)$  denotes its symbol, we define its quasi-homogeneous principal symbol  $P(x, \xi)$  as

$$P(x, \xi) = \lim_{r \rightarrow +\infty} r^{-1} p(r^{1/4k} x, r^{1/4} \xi),$$

following [23].

Then we have

$$\begin{aligned} P(x, \xi) &= |\xi^2 + e^{i\theta} x^{2k}|^2 = \xi^4 + 2 \cos \theta \xi^2 x^{2k} + x^{4k}, \\ P(r^{1/4k} x, r^{1/4} \xi) &= r P(x, \xi), \quad r > 0. \end{aligned} \tag{4.2}$$

Moreover  $P$  is globally elliptic, in the sense that

$$\forall (x, \xi) \neq (0, 0), \quad |P(x, \xi)| > 0. \tag{4.3}$$

Hence the results of [23] allow us to apply the following Weyl formula:

$$N(t) := \#\{j \geq 1 : \nu_j \leq t\} \underset{t \rightarrow +\infty}{\sim} \int_{P(x, \xi) \leq t} dx d\xi,$$

which, with  $t = \nu_n$  and using (4.2), yields

$$n \underset{n \rightarrow +\infty}{\sim} C \nu_n^{\frac{k+1}{4k}}$$

where  $C = \text{Vol } P^{-1}([0, 1])$ .

Thus the series  $\sum \nu_n^{-p/2}$  converges if and only if

$$\sum_{n=1}^{+\infty} n^{-\frac{2kp}{k+1}} < +\infty,$$

that is if and only if  $p > \frac{k+1}{2k}$ . □

Since the operator  $\mathcal{A}(2k, \theta)$  is sectorial and its numerical range is included in the sector  $\mathcal{S}_\theta = \arg^{-1}[0, \theta]$ , the resolvent estimate

$$\|(\mathcal{A}(2k, \theta) - \lambda)^{-1}\| = \mathcal{O}(|\lambda|^{-1}) \quad (4.4)$$

holds outside  $\mathcal{S}_\theta$ , and if we denote  $p = \frac{k+1}{2k} + \varepsilon$ , then

$$\theta < \frac{(k+1)\pi}{2k} < \frac{\pi}{\frac{k+1}{2k} + \varepsilon} = \frac{\pi}{p} \quad (4.5)$$

for  $\varepsilon$  small enough, as soon as  $k > 1$ .

Consequently, Theorem 1.4 follows from Lemma 4.1 and Corollary 31 of [10], p. 1115.

In the next paragraph, we prove Corollary 1.5.

## 4.2 Semigroup decomposition and control of the remainder

Theorems 1.1 and 1.2, for  $m = 2k$ , with  $k = 1/2$  or  $k \geq 1$ , yield as  $n \rightarrow +\infty$

$$\begin{aligned} \|e^{-t\lambda_n} \Pi_n\| &= e^{-t \operatorname{Re} \lambda_n} \kappa_n(m, \theta) \\ &= \frac{K(m, \theta)}{\sqrt{n}} e^{-t \operatorname{Re} \lambda_n} e^{c_k(\theta)|\lambda_n|^{\frac{k+1}{2k}}} (1 + o(1)) \\ &= \frac{K(m, \theta)}{\sqrt{n}} e^{-t|\lambda_n| \cos \frac{\theta}{k+1}} e^{c_k(\theta)|\lambda_n|^{\frac{k+1}{2k}}} (1 + o(1)) \end{aligned} \quad (4.6)$$

where, in the case  $m = 1$ , the constants  $K(1, \theta) = K(\theta)$  and  $c_{1/2}(\theta) = C(\theta)$  are those of Theorem 1.1.

Thus, for  $k = \frac{1}{2}$  and for any  $t > 0$ , there exists  $N_t \in \mathbb{N}$  such that

$$\forall n \geq N_t, \quad e^{-t|\lambda_n| \cos \frac{2\theta}{3}} e^{c_{1/2}(\theta)|\lambda_n|^{\frac{3}{2}}} \geq e^{\frac{c_{1/2}(\theta)}{2}|\lambda_n|^{\frac{k+1}{2k}}},$$

and (4.6) is the general term of a strongly divergent series.

For  $k = 1$ , (4.6) yields

$$\|e^{-t\lambda_n} \Pi_n\| = \frac{K(2, \theta)}{\sqrt{n}} e^{(c_1(\theta) - t \cos \frac{\theta}{2})|\lambda_n|} (1 + o(1)),$$

hence the series  $\Sigma_2(t)$  is norm-convergent  $t > T$  and not norm-convergent for  $t < T$ .

Lastly, if  $k \geq 2$ , there exists  $N_t$  such that

$$\forall n \geq N_t, \quad e^{-t|\lambda_n| \cos \frac{\theta}{k+1}} e^{c_k(\theta)|\lambda_n|^{\frac{k+1}{2k}}} \leq e^{|\lambda_n|(-\frac{t}{2} \cos \frac{\theta}{k+1})},$$

and (4.6) is controlled by the general term of a convergent series.

To check that the series  $\Sigma_m(t)$  (when convergent) converges towards the semi-group associated with  $\mathcal{A}(m, \theta)$ , we use the density of the family  $(u_n)$ , where the eigenfunctions  $u_n$  are assumed to be normalized by the condition  $\langle u_n, \bar{u}_n \rangle = 1$ , so that  $(u_n, \bar{u}_n)_{n \geq 1}$  is a biorthogonal family (see [6]), namely

$$\forall n, m \in \mathbb{N}, \quad \langle u_n, \bar{u}_m \rangle = \delta_{n,m}. \quad (4.7)$$

Then we have

$$e^{-t\mathcal{A}(m, \theta)} u_n = e^{-t\lambda_n} u_n$$

and on the other hand,

$$\Sigma_m(t) u_n = \sum_{j=1}^{+\infty} e^{-t\lambda_j} \Pi_j u_n = e^{-t\lambda_n} u_n.$$

Here we used the formula

$$\Pi_j f = \langle f, \bar{u}_j \rangle u_j$$

(see [6], [4]) which holds for rank 1 spectral projections, together with the biorthogonal property (4.7).

Hence by linearity,  $e^{-t\mathcal{A}(m, \theta)}$  and  $\Sigma_m(t)$  coincide on  $\text{Vect}\{u_n : n \geq 1\}$ , and hence on  $\mathcal{D}(\mathcal{A}(m, \theta))$  by density (see theorem 1.4).

To prove the inequality on the remainder, we write  $e^{-t\mathcal{A}(m, \theta)}(I - \Pi_{<N})$  as the remainder of order  $N$  of the series  $\Sigma_m(t)$ , which we estimate roughly to get (1.15).

## A The complex WKB method

Here we recall the main theorem used in section 3. The general theory of complex WKB method is detailed in [21], [29], [12], and we will follow the point of view of [21].

Let us first recall the following convention: for  $\beta \in \mathbb{R}$  and  $x \in \mathbb{C}$ , a *path joining*  $+\infty e^{i\beta}$  to  $x$  will denote a path  $\gamma : ]-\infty, t_0] \rightarrow \mathbb{C}$  of the complex plane such that  $\gamma(t_0) = x$  and such that  $\gamma(]-\infty, T])$  coincide with the ray  $[R, +\infty[ee^{i\beta}$ , for some  $T < t_0$  and some  $R > 0$ .

We study the asymptotic behavior as  $|x| \rightarrow +\infty$  or  $h \rightarrow 0$  of the solutions  $\psi_h$  of equation

$$-h^2 \frac{d^2}{dx^2} \psi_h(x) + f(x) \psi_h(x) = 0, \quad x \in \mathbb{C}, \quad (A.1)$$

where  $f$  is an holomorphic function on  $\mathbb{C}$ . Let  $\mathcal{D}$  be a simply connected subdomain of  $\mathbb{C} \setminus f^{-1}(0)$ , and consider a determination of the function  $\sqrt{f}$  on  $\mathcal{D}$ . For a fixed  $a_0 \in \mathcal{D}$  (or  $a_0$  a zero of  $f$ ), we put

$$S(z) := \int_{a_0}^z \sqrt{f(t)} dt, \quad (A.2)$$

where the integral is taken along any path defined on  $\mathcal{D}$ .

Let  $a_1 \in \mathbb{C}$ , with possibly  $a_1 = +\infty e^{i\beta}$ ,  $\beta \in \mathbb{R}$ . We call *canonical path* joining  $a_1$

to  $z$ , and we denote by  $\gamma(a_1, z)$ , a path of the complex plane joining  $a_1$  to  $z$  such that the function  $\operatorname{Re} S \circ \gamma(a_1, z)$  is decreasing. We then denote

$$\Delta(a_0, a_1) = \{z \in \mathbb{C} : \exists \gamma(a_1, z) \text{ canonical path}\},$$

and for  $\varepsilon > 0$ ,

$$\Delta_\varepsilon(a_0, a_1) = \{z \in \Delta(a_0, a_1) : d(z, \partial\Delta(a_0, a_1)) \geq \varepsilon\},$$

where  $\partial\Delta(a_0, a_1)$  denotes the boundary of  $\Delta(a_0, a_1)$ .

We also define the function

$$\sigma := \frac{1}{f^{3/4}} \left[ \frac{1}{f^{1/4}} \right]'' . \quad (\text{A.3})$$

Then the following theorem holds, see [21], Theorem 11.1 (p. 222) and Chapter 10.

**Theorem A.1 (WKB estimate)** *Let  $f$ ,  $a_0$ ,  $a_1$ ,  $\Delta(a_0, a_1)$  and  $\Delta_\varepsilon(a_0, a_1)$  be as above, and  $\Sigma \subset \Delta_\varepsilon(a_0, a_1)$  (not necessarily bounded). If  $a_1 = +\infty e^{i\beta}$ , we assume that  $\operatorname{Re} S(y e^{i\beta}) \rightarrow +\infty$  as  $y \rightarrow +\infty$ . We also assume that there exist positive constants  $k$ ,  $\rho$  and  $M$  such that, for all  $z \in \Delta(a_0, a_1)$ ,*

$$|\sigma(z)| \leq \frac{k}{1 + |S(z)|^{1+\rho}}, \quad (\text{A.4})$$

and for all  $x \in \Sigma$ ,

$$\int_{\gamma(a_1, z)} \frac{|S'(x)|}{1 + |S(x)|^{1+\rho}} dx \leq M, \quad (\text{A.5})$$

where  $\gamma(a_1, z)$  is a canonical path joining  $a_1$  to  $z$ .

Then, there exists an analytic solution  $\psi_h$  in  $\mathbb{C}$  of equation (A.1) such that:

1. For all  $\alpha \in \mathbb{R}$  satisfying

$$\exists R_\alpha > 0, \quad [R_\alpha, +\infty[ e^{i\alpha} \subset \Sigma,$$

we have

$$\psi_h(z) = \frac{1}{f(z)^{1/4}} \exp\left(-\frac{1}{h} S(z)\right) (1 + o(1)), \quad (\text{A.6})$$

uniformly with respect to  $h$ , as  $z$  goes to infinity in direction  $\arg^{-1}\{\alpha\}$ . Moreover, if  $\operatorname{Re} S(y e^{i\alpha}) \rightarrow +\infty$  as  $y \rightarrow +\infty$ , then  $\psi_h$  is the unique solution of (A.1) satisfying (A.6), and for any solution  $\tilde{\psi}_h \in L^2(\mathbb{R}^+ e^{i\alpha})$ , there exists  $c \in \mathbb{C}$  such that  $\tilde{\psi}_h = c\psi_h$ .

2. There exist functions  $u_j = u_j(a_0, a_1)$ ,  $j \geq 1$ , such that

$$\psi_h(z) \underset{h \rightarrow 0}{\sim} \frac{1}{f(z)^{1/4}} \left( 1 + \sum_{j=1}^{+\infty} u_j(z) h^j \right) \exp\left(-\frac{1}{h} S(z)\right) \quad (\text{A.7})$$

uniformly with respect to  $z$  in  $\Sigma$ .

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